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Incentive Compatibility and Differentiability New Results and Classic Applications

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Abstract. We provide several generalizations of Mailath's (1987) result that in games of asymmetric information with a continuum of types incentive compatibility plus separation implies differentiability of the informed agent's strategy. The new results extend the theory to classic models in finance such as Leland and Pyle (1977), Glosten (1989), and DeMarzo and Duffie (1999), that were not previously covered.

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1 Introduction

In many problems of asymmetric information, one agent has private information upon which she bases her actions, and uninformed agents act based on inferences from these actions. Because the informed agent can reveal information through her actions, she chooses her actions strategically. If the informed agent's action as a function of her private information is one-to-one, then her strategy is said to be *separating* and her actions completely reveal her private information.

If the agent's private information (her "type") is given by a continuously distributed real-valued random variable, incentive-compatible separating strategies in such interactions can easily be characterized by a differential equation, *if* the strategy is known to be differentiable. But exactly because the strategy is not known, differentiability cannot be taken for granted. This poses a serious problem for the determination and uniqueness of equilibrium.

In many cases, however, differentiability is an implication of incentive-compatibility. For a large class of signaling games and related settings, Mailath (1987) has shown that any incentive-compatible separating strategy of the informed agent must be differentiable and hence satisfy the standard differential equation. Unfortunately, the assumptions in Mailath (1987) rule out many important applications. In particular, as we describe below, they do not cover the models of Leland and Pyle (1977), Glosten (1989), and DeMarzo and Duffie (1999) that are at the core of modern theories of corporate finance and market microstructure.

In this paper, we provide appropriate generalizations of Mailath (1987) to cover these models. The new results can be grouped into two categories. First, we provide new sufficient conditions on payoff functions for differentiability to be implied by incentive-compatibility. For example, we show that differentiability can obtain even in linear models, which are not covered by Mailath (1987). This extends the analysis to those models in corporate finance or Industrial Organization that use risk-neutrality, where the classic first-order conditions of expected utility theory do not apply.

The second category of results refers to the underlying type and action sets. We show that the original and our new sufficiency conditions extend to non-compact (in particular, unbounded) type sets, and to bounded action sets (in Mailath (1987) the action set is $\mathbb{R}$). This is important for two reasons. First, many applications, in particular in finance, naturally involve unbounded type sets, for example, when using normally distributed returns, or bounded action sets, for example when short-selling is not allowed. Second, working with arbitrary type sets makes it possible to apply the sufficient conditions locally instead of globally. For this, one simply considers subintervals of the original type set, when the sufficient conditions do not hold globally (because, for example, a derivative vanishes somewhere), but do hold locally (because the derivative cannot vanish everywhere). We provide an example in which global differentiability can be shown by "patching together" local arguments.

Our interest in the differentiability of separating strategies leads us to study environments with differentiable payoffs. The standard theory of ordinary differential equations, together with the usual boundary conditions then implies the uniqueness of separating equilibrium strategies. This uniqueness also holds more generally in environments with continuous payoffs. See Roddie (2011) for global conditions that guarantee uniqueness of the separating equilibrium when there is a continuum of types.

2 The Model

An informed agent knows the state of nature $\omega \in \Omega \subset \mathbb{R}$ and one or more uninformed agents react to the informed agent's action $x \in \mathcal{X} \subset \mathbb{R}$ on the basis of inferences drawn from x about ω. The sets Ω and $\mathcal{X}$ are intervals; they may be bounded or unbounded, and we do not require the intervals to be open or closed. For our purposes, this interaction can be summarized by the $\mathcal{C}^2$ function

$$\begin{array}{ccc} V : \Omega^2 \times \mathcal{X} & \longrightarrow & \mathbb{R}, \\ (\omega, \widehat{\omega}, x) & \longmapsto & V(\omega, \widehat{\omega}, x), \end{array} \tag{1}$$

denoting the informed agent's payoff from taking action x when the true state of nature is ω and the uninformed agents believe it is $\widehat{\omega}$. We assume V has a $\mathcal{C}^2$-extension (also denoted V) to the closure of $\Omega^2 \times \mathcal{X}$.[1]

As an example, consider the canonical signaling game (Spence (1973); Cho and Kreps (1987)). There is an informed agent who, knowing ω, chooses an action (or costly message) x, followed by an uninformed agent who observing x but not ω, chooses a response $r \in \mathcal{R} \subset \mathbb{R}$. The informed agent's payoff is given by $v(x, r, \omega)$. Given x and a point belief $\widehat{\omega} \in \Omega$ after observing x, denote by $\rho(x, \widehat{\omega})$ a best response for the uninformed player. The informed agent's payoff, given the uninformed agent's best response ρ, can then be written as

$$V(\omega, \widehat{\omega}, x) \equiv v(x, \rho(x, \widehat{\omega}), \omega),$$

[1]For values $(\omega, \widehat{\omega}, x)$ on the boundary of $\Omega^2 \times \mathcal{X}$, the derivatives are the appropriate one-sided derivatives.

which is of the form assumed in (1) if v and ρ are twice continuously differentiable.

We study interactions in which the informed agent's information is fully revealed (as in separating equilibria in signalling games): The informed agent's action is given by a *one-to-one* function $X : \Omega \to \mathcal{X}$, so that $\omega \neq \omega'$ implies $X(\omega) \neq X(\omega')$. Furthermore, X must be *incentive-compatible*, which means that the informed agent finds it optimal to follow this strategy when she knows ω:

$$X(\omega) \in \arg\max_{x \in X(\Omega)} V(\omega, X^{-1}(x), x). \tag{IC}$$

The following assumptions adopt Mailath's (1987) local concavity conditions (4) and (5) to our setting.[2] We will use them to prove some, but not all generalizations of Mailath's (1987) theorem in Section 4.

Assumption 1 *The first-best contracting problem (the problem under full information),*

$$\max_{x \in \mathcal{X}} V(\omega, \omega, x)$$

has a unique solution for all $\omega \in \Omega$, denoted $X^{FB}(\omega)$, for all $\omega \in \Omega$.

Note that if $\mathcal{X}$ is compact the first-best may lie on the boundary of $\mathcal{X}$. We denote the interior of a set A by $\mathrm{int}(A)$.

Assumption 2 *1. For all $\omega \in \mathrm{int}(\Omega)$, $V_{33}(\omega, \omega, X^{FB}(\omega)) < 0$.*

2. There exists $k > 0$ such that for all $(\omega, x) \in \Omega \times \mathcal{X}$,

$$V_{33}(\omega, \omega, x) \geq 0 \Rightarrow |V_3(\omega, \omega, x)| > k.$$

Note that Assumption 1 only implies $V_{33}(\omega, \omega, X^{FB}(\omega)) \leq 0$ for interior ω. The strengthening to Assumption 2.1 is needed in the proof of Lemma B in the appendix. Assumption 2.2 is weaker than strict concavity but stronger than strict quasi-concavity of $V(\omega, \omega, \cdot)$.

The following theorem is the central result of Mailath (1987).

Theorem 1 (Mailath (1987)) *Let $\Omega = [\omega_1, \omega_2]$ and $\mathcal{X} = \mathbb{R}$ and let X be one-to-one and incentive-compatible. Suppose Assumptions 1 and 2 hold, and $V_{13}(\omega, \widehat{\omega}, x) \neq 0$ and $V_2(\omega, \widehat{\omega}, x) \neq 0$ for all $(\omega, \widehat{\omega}, x) \in \Omega^2 \times \mathcal{X}$.*

[2] Since Mailath (1987) took $\mathbb{R}$ as the action space, while we allow for arbitrary real intervals, the assumption that the first order condition $V_3 = 0$ has a unique solution has been replaced with the more general requirement that the first-best contracting problem has a unique solution.

1. *If* $V_3(\omega, \widehat{\omega}, X(\widehat{\omega}))/V_2(\omega, \widehat{\omega}, X(\widehat{\omega}))$ *is a strictly monotone function of* ω *for all* $\widehat{\omega}$*, then* X *is differentiable in the interior of* Ω*,* $\mathrm{int}(\Omega)$*.*

2. (a) *If* $X(\omega_1) = X^{FB}(\omega_1)$ *and* $V_2(\omega, \widehat{\omega}, x) > 0$ *for all* $(\omega, \widehat{\omega}, x) \in \Omega^2 \times \mathcal{X}$*, then* X *is differentiable on* $\Omega \setminus \{\omega_1\}$*.*

   (b) *If* $X(\omega_2) = X^{FB}(\omega_2)$ *and* $V_2(\omega, \widehat{\omega}, x) < 0$ *for all* $(\omega, \widehat{\omega}, x) \in \Omega^2 \times \mathcal{X}$*, then* X *is differentiable on* $\Omega \setminus \{\omega_2\}$*.*

   If X *is differentiable, then it satisfies the differential equation*

$$X'(\omega) = -\frac{V_2(\omega, \omega, X(\omega))}{V_3(\omega, \omega, X(\omega))}. \tag{DE}$$

The differential equation (DE) is a trivial consequence of the incentive constraint (IC), which yields the first-order condition $V_2 + X'V_3 = 0$, *given* differentiability.

The assumption that V_2 never equals zero, and so never changes sign ("belief monotonicity") implies that the direction of desired belief manipulation by the informed agent is unambiguous: if $V_2 > 0$, she benefits from the uninformed side believing her to be of a higher type (respectively, of a lower type if $V_2 < 0$). The assumption that V_{13} never changes sign ("type monotonicity") means that the informed agent's marginal utility from x is monotone in her type. Neither assumption need be satisfied in standard examples, as the following section shows.

The condition that V_3/V_2 is a strictly monotone function of ω for all $(\widehat{\omega}, X(\widehat{\omega}))$ is a weak form of single crossing; we discuss the role of the single crossing property in Section 5 when we introduce Theorem 6.

For signaling games satisfying standard monotonicity properties, the initial value condition pinning down the value of X at either ω_1 or ω_2 in parts 2a and b of Theorem 1 is a simple consequence of sequential rationality:[3] Suppose $V_2 > 0$. Then $\hat{\omega} = \omega_1$ is the worst belief the uninformed agents can have about the informed agent. It is then immediate that in any Nash equilibrium with X separating, if $X^{FB}(\omega_1) \in X(\Omega)$ then $X(\omega_1) = X^{FB}(\omega_1)$.[4] On the other hand, if $X^{FB}(\omega_1) \notin X(\Omega)$, then in response to a deviation

[3] For signaling games with finite type and action spaces, sequential rationality is formalized as sequential equilibrium (Kreps and Wilson, 1982), and with infinite type and action spaces, by various versions of perfect Bayes equilibrium.

[4] Suppose $X^{-1}(X^{FB}(\omega_1)) = \omega' \neq \omega_1$. Then,

$$V(\omega_1, \omega', X^{FB}(\omega_1)) > V(\omega_1, \omega_1, X^{FB}(\omega_1)) > V(\omega_1, \omega_1, X(\omega_1)),$$

and so X is not incentive compatible, a contradiction.

to the action $x = X^{FB}(\omega_1)$, sequential rationality requires the uninformed agents to choose a best reply to *some* belief, and $V_2 > 0$ again implies that ω_1 has a profitable deviation. Finally, note that since $V_3(\omega, \omega, X^{FB}(\omega)) = 0$ when $\mathcal{X} = \mathbb{R}$, (DE) implies $|X'(\omega)| \to \infty$ as $\omega \to \omega_k$ for any ω_k satisfying $X(\omega_k) = X^{FB}(\omega_k)$, and so X cannot be differentiable at the indicated endpoints in parts 2a and b.

3 Three Examples

3.1 Equity Issues

The classic model of equity issues of Leland and Pyle (1977) considers an owner of a firm who wants to raise funds on the stock market by selling her holdings. The uninformed side of the market is "the stock market": a large group of equally informed and well-diversified investors. Investors are willing to invest if in expectation they earn the risk-free rate, normalized to 0.

The company is worth $\omega + \varepsilon$ in the future, where $\omega \in \Omega = [\omega_1, \infty)$ is a positive number and ε is a zero-mean random variable defined on an interval $[\underline{\varepsilon}, \overline{\varepsilon}]$. The expected value of the firm, therefore, is ω. The owner has personal wealth (outside the firm) of w_0 and is risk-averse, with an increasing, strictly concave, twice continuously differentiable money utility function U. The capital market is risk-neutral. The owner considers diversifying his risk by selling a fraction $1 - x$ of the firm in exchange for a payment of t by the capital market.

The owner's utility from an allocation $(x, t) \in [0, 1] \times \mathbb{R}$ is

$$E_\varepsilon U(x(\omega + \varepsilon) + w_0 + t) \tag{2}$$

and that of the capital market (using risk-neutrality)

$$(1 - x)\omega - t. \tag{3}$$

Because the owner is risk-averse, the first-best is $X^{FB}(\omega) = 0$, i.e., to sell the firm completely, regardless of ω. The interaction between the two sides of the market is given by a signaling game in which the owner, knowing the value of ω, proposes an equity issue (x, t) which the stock market accepts or rejects.

As is well known, this game has a large number of equilibria. The literature usually considers equilibria with (i) maximum information transmission that (ii) leave zero expected profits to the market conditional on each type. Property (i) restricts attention to strategies $(X, T) : \Omega \to [0,1] \times \mathbb{R}$ that are one-to-one (fully separating), while property (ii) implies transfers $T(\widehat{\omega}) = (1-x)\widehat{\omega}$, where $\widehat{\omega}$ is the inferred expected value of the firm. One can then ignore $t = T(\widehat{\omega})$ in the analysis and denote a strategy of the informed player (the owner) by $X(\omega)$.

The payoff function V of the informed agent as defined in (1) is

$$V(\omega, \widehat{\omega}, x) = E_\varepsilon U(w_0 + x(\omega + \varepsilon) + (1-x)\widehat{\omega}). \tag{4}$$

We have

$$V_2(\omega, \widehat{\omega}, x) = (1-x) E_\varepsilon U'(w_0 + x(\omega + \varepsilon) + (1-x)\widehat{\omega}),$$

and

$$\begin{aligned} V_{13}(\omega, \widehat{\omega}, x) = {} & E_\varepsilon U'(w_0 + x(\omega + \varepsilon) + (1-x)\widehat{\omega}) \\ & + x(\omega - \widehat{\omega}) E_\varepsilon U''(w_0 + x(\omega + \varepsilon) + (1-x)\widehat{\omega}). \end{aligned}$$

Simple examples show that in general V_{13} can be 0, violating type-monotonicity. Furthermore, $V_2(\omega, \widehat{\omega}, 1) = 0$, violating belief-monotonicity for $x = 1$. Finally, Ω is not compact, and $\mathcal{X}$ is not $\mathbb{R}$.

3.2 Market Microstructure

Motivated by Glosten (1989), consider a market for a risky asset in which risk neutral market makers provide liquidity to an informed risk-averse investor who, depending on her private information, may wish to buy or sell. Let $x \in \mathbb{R}$ denote the quantity of the risky asset traded by the investor, with $x > 0$ corresponding to a purchase and $x < 0$ to a sale. The corresponding monetary transfer from the investor to the market maker is denoted by $t \in \mathbb{R}$; if $t < 0$, $-t$ is the amount received by the investor. If as in standard market microstructure theory, p is the price of the asset, then $t = px$.

The final value of the risky asset is $\nu = s + \varepsilon$. The investor privately observes s and her endowment θ of the risky asset before trade takes place. The random variables (s, θ) describe the investor's private information, and after a trade (x, t), the investor's final wealth is $(x + \theta)(s + \varepsilon) - t$. Under suitable assumptions,[5] the private information of the investor can be summarized by a one-dimensional type ω, whose support Ω may be bounded,

[5] See Mailath and Nöldeke (2008) for details. These assumptions are the same as in Glosten (1989), with the exception that the random variables describing the investor's private information are *not* required to be normally distributed. If ω is normally distributed, then the support Ω of its distribution is $\mathbb{R}$, but in general Ω can be bounded.

and the investor's preferences over trade-transfer pairs are described by the utility function

$$\mathcal{U}(x, t \mid \omega) = b\omega x - rx^2/2 - t, \tag{5}$$

where $b > 1$ and $r > 0$ are two parameters reflecting the investor's risk aversion and characteristics of the underlying stochastic environment. Market makers are risk neutral and maximize expected trading profits. It suffices to consider aggregate trading profits $t - \nu x$. Conditional on ω, expected aggregate trading profits are given by

$$\mathcal{V}(x, t \mid \omega) = t - \omega x. \tag{6}$$

The strategic interaction between the two sides of the market is modeled as a signaling game in which the market makers compete for the investor's trade after observing the posted quantity x. Each market maker i offers a transfer $T_i(x)$ to the informed investor, and the investor then chooses a market maker.

We again consider outcomes with maximum information transmission, i.e., trading schedules that are separating with respect to ω. Competition between market makers implies that if there exists a separating equilibrium, each market maker i must make zero expected profits on each type. By (6), this means

$$T_i(X(\omega)) = \omega X(\omega) \quad \text{for all } \omega \in \Omega \text{ and all } i. \tag{7}$$

Hence, the trading schedule schedule pins down the pricing schedule.

By (5), the payoff function V of the informed investor as defined in (1) is then

$$V(\omega, \widehat{\omega}, x) = (b\omega - \widehat{\omega})x - rx^2/2. \tag{8}$$

We have

$$V_2(\omega, \widehat{\omega}, x) = -x, \tag{9}$$

$$V_{13}(\omega, \widehat{\omega}, x) = b, \tag{10}$$

$$\text{and} \qquad \frac{d}{d\omega}\left\{\frac{V_3(\omega, \widehat{\omega}, x)}{V_2(\omega, \widehat{\omega}, x)}\right\} = -\frac{b}{x}. \tag{11}$$

Equations (9) and (11) violate the assumptions of Theorem 1. Furthermore, while it is possible in the equity issue model of the previous subsection to restrict Ω arbitrarily to a compact interval, the case of an unbounded Ω is important in this case (arising, for example, when ω is normal).[6]

[6] Mailath and Nöldeke (2008) do not prove that separating trading schedules are differentiable. The results here support their assertion that the arguments in Mailath (1987) apply.

3.3 Security Design

The fundamental question in corporate finance is how to allocate the cash flow generated by a firm's assets among its different providers of capital. DeMarzo and Duffie (1999) have argued that this problem should be analyzed in two steps. First, the firm's owners or managers design the security, and second the security is sold to investors. Since the second step may take place significantly later than the first, the firm may have obtained private information concerning the security's payoff once it sells the security. For this second step, DeMarzo and Duffie (1999) therefore consider the following game.

The security has an expected payoff $\omega \in \Omega$, where ω is private information of the firm and $\Omega \subset \mathbb{R}$ is a potentially unbounded interval with left endpoint ω_1. The firm considers selling a quantity $x \in [0, 1]$ of the security to market investors. There are gains from trade because the firm discounts the security's cash flows at a higher rate than the market. Let $\delta < 1$ be the firm's discount rate relative to that of the market (which is normalized to 1). The firm and the market are both risk-neutral. If the firm sells the amount x of the security for a total of t, the firm's payoff is

$$t + (1 - x)\delta\omega \tag{12}$$

and the market investors' payoff is

$$x\omega - t. \tag{13}$$

Market investors are competitive and must make zero expected profits for each value of ω. Hence, if they believe the expected value of the security to be $\widehat{\omega}$, they will pay $t = x\widehat{\omega}$. Inserting this into (12) yields the payoff function V of the informed investor as defined in (1):

$$\begin{aligned} V(\omega, \widehat{\omega}, x) &= x\widehat{\omega} + (1 - x)\delta\omega \\ &= \delta\omega + (\widehat{\omega} - \delta\omega)x. \end{aligned}$$

The informed agent's payoff function V is linear in x and therefore violates Assumption 2 of Theorem 1. Furthermore, Ω need not be compact, though $\mathcal{X}$ is.

4 The Generalized Theorems

In this section, we provide a set of results that significantly expand the applicability of Theorem 1. The proofs of the results, which recycle Mailath's original proof and add a number of new elements, are in the appendix.

We generalize Theorem 1 in several respects. First we generalize the theorem by weakening its assumptions on type and action sets. As stated in Section2, Ω and $\mathcal{X}$ can now be arbitrary real intervals. With respect to Ω, the argument is not immediate because Mailath's proof uses uniform convergence (for which compactness is needed) and exploits the behavior of X on the boundary of Ω. With respect to $\mathcal{X}$, the difficulty is that the equivalence of $V_3(\omega, \omega, X(\omega)) = 0$ and $X(\omega) = X^{FB}(\omega)$ breaks down for $X(\omega)$ on the boundary of $\mathcal{X}$.

We also weaken Mailath's (1987) assumption that the conditions on the payoff function must hold for all $(\omega, \widehat{\omega}, x) \in \Omega^2 \times \mathcal{X}$. The necessary changes in Mailath's proof are simple, but the new generality is useful because often there is some a priori information about the graph of X that makes it easier to verify the conditions only on the relevant subsets of $\Omega^2 \times \mathcal{X}$. The equity issue example in Section 6.1 is an example.

These observations yield the following generalization of Theorem 1.

Theorem 2 *Let X be one-to-one and incentive-compatible. Suppose Assumptions 1 and 2 hold and $V_{13}(\omega, \omega, x) \neq 0$ for all $(\omega, x) \in \mathrm{int}(\Omega) \times \mathcal{X}$.*

1. *Suppose $V_2(\omega, \widehat{\omega}, X(\widehat{\omega})) \neq 0$ for all $\omega, \widehat{\omega} \in \mathrm{int}(\Omega)$, and*

$$V_3(\omega, \widehat{\omega}, X(\widehat{\omega}))/V_2(\omega, \widehat{\omega}, X(\widehat{\omega}))$$

   is a strictly monotone function of ω for all $\widehat{\omega} \in \mathrm{int}(\Omega)$. Then X is differentiable on $\mathrm{int}(\Omega)$.

2. (a) *Assume that $\Omega = [\omega_1, \omega_2]$ or $\Omega = [\omega_1, \infty)$ and that $X(\omega_1) = X^{FB}(\omega_1)$. If $V_2(\omega, \omega, X(\omega)) > 0$ for all $\omega \in \Omega$ then X is differentiable on $\Omega \setminus \{\omega_1\}$.*

   (b) *Assume that $\Omega = [\omega_1, \omega_2]$ or $\Omega = (-\infty, \omega_2]$ and that $X(\omega_2) = X^{FB}(\omega_2)$. If $V_2(\omega, \omega, X(\omega)) < 0$ for all $\omega \in \Omega$ then X is differentiable on $\Omega \setminus \{\omega_2\}$.*

At all points of differentiability, X satisfies the differential equation (DE).

Theorem 2.2 clarifies the role of the boundary conditions in Theorem 1 and shows that only one boundary condition is necessary to obtain the result. This extends the validity of the theorem to the case of intervals that are either unbounded from below or from above. Theorem 2.1 is the same statement as in Theorem 1 but without the restrictions discussed above. The comparison of Theorem 2.1 and Theorem 2.2 therefore extends and clarifies

Mailath's (1987) observation that in order to prove differentiability one can use single crossing or a boundary condition.[7]

We now consider sufficient conditions that are different in spirit from Theorem 1. Our first such result is that incentive-compatibility implies differentiability in models with strictly monotone payoffs and compact choice sets, as in DeMarzo and Duffie (1999).

Theorem 3 *Let X be one-to-one and incentive-compatible, and suppose that $\mathcal{X}$ is compact. For any $\omega \in \Omega$, if $V_3(\omega,\omega,x) \neq 0$ for all $x \in \mathcal{X}$, then X is differentiable at ω. At all points of differentiability, X satisfies the differential equation (DE).*

The assumption on V_3 in Theorem 3 is slightly stronger than strict monotonicity in x. Note that if $V(\omega,\omega,\cdot)$ is strictly monotone in x, $X^{FB}(\omega)$ exists and is unique (hence, Assumption 1 is satisfied) and lies on the boundary of $\mathcal{X}$. As noted earlier, if ω is on the boundary of Ω or x on the boundary of $\mathcal{X}$, the derivatives are the relevant one-sided derivatives.

Corollary 1 *Let X be one-to-one and incentive-compatible. If V is affine in $x \in \mathcal{X}$,*

$$V(\omega,\widehat{\omega},x) = A(\omega,\widehat{\omega}) + B(\omega,\widehat{\omega})x, \tag{14}$$

with $B(\omega,\omega) \neq 0$ for all $\omega \in \Omega$, then X is differentiable at every $\omega \in \Omega$ and satisfies the linear differential equation

$$B(\omega,\omega)X'(\omega) + B_2(\omega,\omega)X(\omega) = -A_2(\omega,\omega). \tag{15}$$

Corollary 1 is a direct consequence of Theorem 3, because (14) implies that $V_3(\omega,\omega,x) = B(\omega,\omega) \neq 0$ for all $\omega \in \Omega$, and (15) is the re-write of (DE) for the affine case.

Corollary 1 is useful because many standard models in corporate finance, as in Industrial Organization, work with linear preferences, which often gives rise to valuations V of the form (14).[8] The theorem is surprising because constrained optimization problems with linear objective functions often yield discontinuous solutions. The assumption that the action set $\mathcal{X}$ is compact does not force the solution to lie on the boundary: the optimal X typically lies in the interior of $\mathcal{X}$. Instead, compactness is needed to prove that for

[7] See Section 5 for further discussion of the single crossing property.

[8] Next to DeMarzo and Duffie (1999), for an example of such a model with with linear preferences and continuous types in the IO/finance area, see Burkart and Lee (2011), who also discuss further papers.

any ω_0 and any sequence $\omega_n \to \omega_0$, $V(\omega_0, \omega_0, X(\omega_n)) \to V(\omega_0, \omega_0, X(\omega_0))$ (Lemma D in the appendix). This is a crucial insight to establish the continuity of X, from which, in turn the differentiability of X can be deduced.

Instead of assuming compactness of $\mathcal{X}$, this insight can also be proved by using our relaxed concavity Assumption 2, which we do in the following theorem.

Theorem 4 *Let X be one-to-one and incentive-compatible. Suppose Assumptions 1 and 2 hold.*

1. *For any $\omega \in \Omega$, if $V_3(\omega, \omega, x) \neq 0$ for all $x \in \mathcal{X}$, then X is differentiable at ω.*

2. *For any $\omega \in \mathrm{int}(\Omega)$, if $V(\omega, \omega, \cdot)$ is a monotone function in x and if $V_2(\omega, \omega, X^{FB}(\omega)) \neq 0$, then X is differentiable at ω.*

3. *Assume that $V_{13}(\omega, \omega, x) \neq 0$ and $V_{12}(\omega, \omega, x) \leq 0$ for all $(\omega, x) \in \mathrm{int}(\Omega) \times \mathcal{X}$. If $V_2(\omega, \omega, X(\omega)) > 0$ or if $V_2(\omega, \omega, X(\omega)) < 0$ for all $\omega \in \mathrm{int}(\Omega)$, then X is differentiable on $\mathrm{int}(\Omega)$.*

At all points of differentiability, X satisfies the differential equation (DE).

Theorems 3 and 4 provide new conditions on V to establish the differentiability of X. As in Theorem 2, the assumptions on the partial derivatives need not hold for all $(\omega, \widehat{\omega}, x) \in \Omega^2 \times \mathcal{X}$.

Theorems 3 and 4.1 are almost identical, differing only in one assumption (Theorem 3 uses the compactness of $\mathcal{X}$, while statement 4.1 uses the quasi-concavity of Assumption 2). Theorem 4.2 addresses the same situation slightly differently, because under Assumption 2 the fact that $V(\omega, \omega, \cdot)$ is a monotone function in x only implies $V_3(\omega, \omega, x) \neq 0$ for almost all $\omega \in \mathrm{int}(\Omega)$ and $x \in \mathrm{int}(\mathcal{X})$.

The condition $V_{12} \leq 0$ ("manipulation monotonicity"), required in Theorem 4.3, is mild and satisfied in almost all examples we know of (though it does fail in Kartik, Ottaviani, and Squintani (2007)). It requires that the informed agent's gain from manipulating the uninformed beliefs upwards does not increase in her type.

Theorems 3 and 4 show that in order to prove differentiability, neither single crossing nor a boundary condition are necessary. Theorems 3, 4.1, and 4.2 are useful because the required monotonicity is often easy to verify, and is structurally novel because it is local (i.e., it only requires conditions at ω_0 to establish differentiability at ω_0). Note, however, that also Theorem

2 and Theorem 4.3 hold for *arbitrary* sets Ω, so the results can be applied "piecemeal" to open subsets $\Omega_0 \subset \Omega$. This is particularly useful if the regularity assumptions for V are known not to hold on the whole domain.

Finally, we report a useful result that follows directly from Mailath's proof:

Theorem 5 *Let X be one-to-one and incentive-compatible. Suppose Assumption 1 holds and that either $\mathcal{X}$ is compact or Assumption 2 holds. If, for any $\omega \in \Omega$, $X(\omega) = X^{FB}(\omega)$, then X is continuous at ω.*

The assumption in Theorem 5 seems difficult to verify a priori, because one needs to know X, which one actually wants to characterize. However, since it is often straightforward to find the $\widehat{\omega}$ for which $X(\widehat{\omega}) = X^{FB}(\widehat{\omega})$, the injectivity of X then implies that one can apply Theorems 2–4 to $\{\omega > \widehat{\omega}\}$ or $\{\omega < \widehat{\omega}\}$, or to $\mathcal{X} \setminus \{X(\widehat{\omega})\}$ if $X(\widehat{\omega})$ is on the boundary of $\mathcal{X}$. Hence, the statement is useful to "fill possible holes" left by the other statements. Subsection 6.2 provides an example for this technique.

5 When Does Differentiability Imply Incentive Compatibility?

Theorems 2, 3, and 4 identify conditions under which incentive-compatibility implies differentiability and so (DE). To complete our discussion we briefly turn to the question under what conditions the converse is true, i.e. when differentiability (in the form of (DE)) implies incentive-compatibility. Under the assumptions of Theorem 1, Mailath (1987, Theorem 3) showed that the converse holds if V satisfies the single-crossing property on the graph of X. The next theorem shows that this statement continues to be true under weaker assumptions. Moreover, the property is also *locally* necessary (the statement of Mailath (1987, Theorem 3) incorrectly describes the locality notion).

The Spence (1973)-Mirrlees (1971) single-crossing property requires the agent's marginal rate of substitution between her action (x) and that of the uninformed agents be appropriately monotone in her type. In our examples, the uninformed agents' action is a monetary transfer, and as is typical, the action is monotone in beliefs about type. Consequently, in our reduced form model, the relevant marginal rate of substitution is between $\widehat{\omega}$ and x, that is, $V_3(\omega, \widehat{\omega}, x)/V_2(\omega, \widehat{\omega}, x)$. The adjective "appropriately" captures the requirement that, for example, in job market signaling, single crossing is implied by more (rather then less) able workers having a lower marginal cost of education. If less able workers have the lower marginal cost of education, then the marginal rate of substitution between education and wage is an *increasing* function of ability. While monotonic, such a marginal rate of substitution precludes the existence of a separating equilibrium.

The single-crossing property imposes a uniform structure on the derivatives of V that implies global optimality from the first-order condition (which is essentially the differential condition (DE)). Conversely, the second-order condition for local optimality implied by incentive-compatibility is essentially the local single-crossing condition.

The following theorem on the role of single crossing is proved in the appendix. Since X' and V_2 do not change sign, (16) and (17) both imply that $V_3(\omega, \widehat{\omega}, x)/V_2(\omega, \widehat{\omega}, x)$ is monotonic in ω (with the signs of X' and V_2 jointly determining the appropriate monotonicity). Condition (16) implies global monotonicity, in that it holds everywhere on the graph of X, while (17) implies local monotonicity, in that it only holds for the derivative evaluated at $\widehat{\omega} = \omega$.

Theorem 6 *Assume that the one-to-one function X is continuous on Ω and satisfies the differential equation (DE) on the interior of Ω. Suppose $V_2(\omega, \widehat{\omega}, X(\widehat{\omega})) \neq 0$ for all $\omega, \widehat{\omega} \in \Omega$.*

1. *If*
$$X'(\omega) V_2(\omega, \widehat{\omega}, X(\widehat{\omega})) \frac{d}{d\omega} \left\{ \frac{V_3(\omega, \widehat{\omega}, X(\widehat{\omega}))}{V_2(\omega, \widehat{\omega}, X(\widehat{\omega}))} \right\} \geq 0 \tag{16}$$
for all $\omega, \widehat{\omega} \in \Omega$, then X is incentive-compatible.

2. *If X is incentive-compatible, then*
$$X'(\omega) V_2(\omega, \omega, X(\omega)) \; \frac{d}{d\omega} \left\{ \frac{V_3(\omega, \widehat{\omega}, X(\widehat{\omega}))}{V_2(\omega, \widehat{\omega}, X(\widehat{\omega}))} \right\} \bigg|_{\widehat{\omega}=\omega} \geq 0 \tag{17}$$
for all $\omega \in \Omega$.

6 The Examples Revisited

6.1 Equity Issues

Since the payoff of the informed party in the model of Section 3.1, V given by (4), is strictly concave in x, one can apply Theorem 4 to obtain differentiability. In fact, by statement 2, X must be differentiable on (ω_1, ∞), and by Theorem 5, it is continuous at ω_1.

6.2 Market Microstructure

From (8), the first-best in the model of Section 3.2 is given by

$$X^{FB}(\omega) = \frac{b-1}{r}\omega.$$

Under any competitive incentive-compatible separating price-quantity schedule, type $\omega = 0$ must get her first-best allocation, $x = 0$.[9] Separation then implies that every other type must choose a non-zero quantity. Hence, $V_2(\omega, \omega, X(\omega)) \neq 0$ for all $\omega \neq 0$ by (9). Since $V_{12} \equiv 0$, Theorem 4.3 therefore implies that any incentive-compatible schedule X must be differentiable on the open sets $(0, \infty)$ and $(-\infty, 0)$.

By Theorem 5, the schedule X is continuous at $\omega = 0$. Calculating the derivatives of X on $(0, \infty)$ and $(-\infty, 0)$ from (DE) shows that the left-hand and right-hand derivative of X at $\omega = 0$ exist and are identical. Hence, X is differentiable on all of Ω.

Although the model of Section 3.2 is a signaling model, the analysis also applies to separating equilibria of competitive screening models. However, care needs to be exercised in applying these results to competitive screening models, as such models often fail to have separating equilibria (Riley, 2001, page 446).[10]

One of the main insights in Glosten (1989) is his non-existence result. In particular, he shows that in the case $b \leq 2$, the differential equation (DE) for $\Omega = \mathbb{R}$ does not have a separating solution (see also Hellwig (1992)). This suggests that too much competition may be detrimental for market activity. This conclusion requires that every equilibrium trading schedule is differentiable, a result missing in Glosten (1989), but which is implied by Theorems 4 and 5.

It is worth noting that this conclusion also requires viewing separation as an implication of competition. Mailath and Nöldeke (2008) argue that competitive pricing does not lead to market breakdown, even if the investor's private information has unbounded support.

6.3 Security Design

Since the firm's payoff function V is linear in x, if the firm's strategy X is incentive-compatible and one-to-one, Theorem 3 implies that it is differen-

[9]Since $V(0, 0, X(0)) \geq 0$ (type $\omega = 0$ has the option of choosing $x = 0$), and $V(0, 0, x) = -rx^2/2$, which is strictly negative if $x \neq 0$, we have $X(0) = 0$.

[10]We thank a referee and associate editor for the Riley (2001) reference and for helping us to appreciate this point.

tiable for all $\omega > 0$ and satisfies the differential equation

$$(1-\delta)\omega X'(\omega) + X(\omega) = 0. \tag{18}$$

It can be easily verified that (18) has the solution

$$X(\omega) = a\omega^{-\frac{1}{1-\delta}}, \tag{19}$$

where $a \geq 0$ is a constant of integration. Since $X(\omega) \in [0,1]$ by construction, (19) implies that $\omega_1 = \inf \Omega$ and a must satisfy $\omega_1 \geq a^{1-\delta}$ for a solution to exist. In particular, if the interaction in the model is a signaling game (as in Section 3 of DeMarzo and Duffie (1999)), then firm type ω_1 must obtain its most preferred allocation $x = 1$, and the constant of integration is

$$a = \omega_1^{\frac{1}{1-\delta}}.$$

Note that this implies that Ω must be bounded away from 0 for a non-trivial solution to exist.[11]

Appendices

A Preliminary Results

In what follows, Ω and $\mathcal{X}$ are intervals of $\mathbb{R}$, and X is a one-to-one function satisfying (IC). As in Mailath (1987), we first conduct some preliminary calculations. Fix $\omega_0 \in \Omega$. Define, for arbitrary $\omega, \widehat{\omega} \in \Omega$ and $x \in \mathcal{X}$,

$$g(\omega, \widehat{\omega}, x) \equiv V(\omega, \widehat{\omega}, x) - V(\omega, \omega_0, X(\omega_0)).$$

Since V is $\mathcal{C}^2$ on the closure of $\Omega^2 \times \mathcal{X}$, so is g. The derivatives are well defined on the boundary of the relevant domains (in which case they are the appropriate one-sided derivatives). Moreover,

$$g(\omega, \omega_0, X(\omega_0)) = g_1(\omega, \omega_0, X(\omega_0)) = 0, \quad \forall \omega. \tag{A.1}$$

Incentive compatibility implies

$$g(\omega_0, \omega, X(\omega)) \leq 0 \tag{A.2}$$

$$\text{and} \quad g(\omega, \omega, X(\omega)) \geq 0. \tag{A.3}$$

[11] DeMarzo and Duffie (1999) derive (19) assuming differentiability and cite an earlier unpublished version of their paper for a proof.

For any $\omega \in \Omega$ and $\lambda \in [0,1]$, define

$$[\omega;\lambda]_1 \equiv (\lambda\omega_0 + (1-\lambda)\omega, \omega, X(\omega))$$

and for any $\mu \in [0,1]$, define

$$[\omega;\mu]_{23} \equiv (\omega_0, \mu\omega_0 + (1-\mu)\omega, \mu X(\omega_0) + (1-\mu)X(\omega)).$$

Expanding $g(\omega,\omega,X(\omega))$ around $(\omega_0,\omega,X(\omega))$ as a second order Taylor expansion yields

$$\begin{aligned} g(\omega,\omega,X(\omega)) = g(\omega_0,\omega,X(\omega)) + g_1(\omega_0,\omega,X(\omega))(\omega-\omega_0) \\ + \tfrac{1}{2}g_{11}([\omega;\lambda]_1)(\omega-\omega_0)^2 \end{aligned}$$

for some $\lambda \in [0,1]$. Next, expanding $g_1(\omega_0,\omega,X(\omega))$ around $(\omega_0,\omega_0,X(\omega_0))$ as a first order Taylor expansion yields

$$\begin{aligned} g_1(\omega_0,\omega,X(\omega)) = g_1(\omega_0,\omega_0,X(\omega_0)) \\ + g_{12}([\omega;\mu]_{23})(\omega-\omega_0) + g_{13}([\omega;\mu]_{23})(X(\omega)-X(\omega_0)) \end{aligned}$$

for some $\mu \in [0,1]$. Because the first term on the right-hand side is 0 by (A.1), combining these two expressions yields

$$\begin{aligned} g(\omega,\omega,X(\omega)) &= g(\omega_0,\omega,X(\omega)) \\ &+ (\omega-\omega_0)\Big\{ \left[\tfrac{1}{2}g_{11}([\omega;\lambda]_1) + g_{12}([\omega;\mu]_{23})\right](\omega-\omega_0) \\ &\qquad + g_{13}([\omega;\mu]_{23})(X(\omega)-X(\omega_0))\Big\}. \end{aligned} \tag{A.4}$$

Expressions (A.2), (A.3), and (A.4) imply

$$\begin{aligned} 0 \geq g(\omega_0,\omega,X(\omega)) \geq -(\omega-\omega_0)\Big\{ \left[\tfrac{1}{2}g_{11}([\omega;\lambda]_1) + g_{12}([\omega;\mu]_{23})\right](\omega-\omega_0) \\ + g_{13}([\omega;\mu]_{23})(X(\omega)-X(\omega_0))\Big\} \end{aligned} \tag{A.5}$$

for some $\lambda, \mu \in [0,1]$.

Lemma A *If X is continuous at ω_0 and $V_3(\omega_0,\omega_0,X(\omega_0)) \neq 0$, then X is differentiable at ω_0 with derivative*

$$X'(\omega_0) = -\frac{V_2(\omega_0,\omega_0,X(\omega_0))}{V_3(\omega_0,\omega_0,X(\omega_0))}.$$

Proof. This is essentially Proposition 2 of Mailath (1987). We include the proof for completeness. Expanding $g(\omega_0, \omega, X(\omega))$ around $(\omega_0, \omega_0, X(\omega_0))$ in (A.5) and dividing by $\omega - \omega_0 > 0$ (for $\omega - \omega_0 < 0$, the following inequalities are reversed, but the argument is unaffected) yields, since $g(\omega_0, \omega_0, X(\omega_0)) = 0$,

$$\begin{aligned}
0 \geq\ & g_2(\omega_0, \omega_0, X(\omega_0)) + \left[\frac{X(\omega) - X(\omega_0)}{\omega - \omega_0}\right] \{g_3(\omega_0, \omega_0, X(\omega_0)) \\
& \quad + \tfrac{1}{2} g_{33}([\omega; \gamma]_{23})(X(\omega) - X(\omega_0) + g_{23}([\omega; \gamma]_{23})(\omega - \omega_0)\} \\
& \qquad + \tfrac{1}{2} g_{22}([\omega; \gamma]_{23})(\omega - \omega_0) \\
\geq & - \left[\tfrac{1}{2} g_{11}([\omega; \lambda]_1) + g_{12}([\omega; \mu]_{23})\right] (\omega - \omega_0) - g_{13}([\omega; \mu]_{23})(X(\omega) - X(\omega_0)),
\end{aligned}$$

for some $\gamma \in [0, 1]$. Since V is $\mathcal{C}^2$ on the closure of $\Omega^2 \times \mathcal{X}$, the cross-partials of V (and so g) are bounded on any compact neighborhood of $(\omega_0, \omega_0, X(\omega_0))$. If X is continuous at ω_0, taking limits as $\omega \searrow \omega_0$ then implies

$$0 \geq g_2(\omega_0, \omega_0, X(\omega_0)) + \lim_{\omega \searrow \omega_0} \left[\frac{X(\omega) - X(\omega_0)}{\omega - \omega_0}\right] g_3(\omega_0, \omega_0, X(\omega_0)) \geq 0,$$

implying the differentiability of X at ω_0 (with the specified value); the derivative is the appropriate one-sided derivative if ω_0 is on the boundary. ■

Under Assumptions 1 and 2, Lemma A implies that X is differentiable at ω if X is continuous there, $X^{FB}(\omega) \in \mathrm{int}(\mathcal{X})$ and $X(\omega) \neq X^{FB}(\omega)$.

Lemma B *Suppose that Assumptions 1 and 2 hold. If X is continuous at $\omega_0 \in \mathrm{int}(\Omega)$, and $V_2(\omega_0, \omega_0, X(\omega_0)) \neq 0$ or $V_2(\omega_0, \omega_0, X^{FB}(\omega_0)) \neq 0$, then $V_3(\omega_0, \omega_0, X(\omega_0)) \neq 0$.*

Proof. Suppose that $V_3(\omega_0, \omega_0, X(\omega_0)) = 0$. Then by Assumption 2.2, $V_{33}(\omega_0, \omega_0, X(\omega_0)) < 0$ (one-sided derivative if $X(\omega_0) \in \{\min \mathcal{X}, \max \mathcal{X}\}$).

Suppose $V_2(\omega_0, \omega_0, X(\omega_0)) > 0$. Expanding $g(\omega_0, \omega, X(\omega))$ around the point $(\omega_0, \omega_0, X(\omega_0))$ in (A.5) and dividing by $\omega - \omega_0 < 0$ yields, since $g(\omega_0, \omega_0, X(\omega_0)) = 0$ and $g_3(\omega_0, \omega_0, X(\omega_0)) = 0$,

$$\begin{aligned}
0 \leq\ & g_2(\omega_0, \omega_0, X(\omega_0)) + \left[\frac{X(\omega) - X(\omega_0)}{\omega - \omega_0}\right] \left\{\tfrac{1}{2} g_{33}([\omega; \gamma]_{23})(X(\omega) - X(\omega_0)\right. \\
& \qquad \left. + g_{23}([\omega; \gamma]_{23})(\omega - \omega_0)\right\} + \tfrac{1}{2} g_{22}([\omega; \gamma]_{23})(\omega - \omega_0) \\
\leq & - \left[\tfrac{1}{2} g_{11}([\omega; \lambda]_1) + g_{12}([\omega; \mu]_{23})\right] (\omega - \omega_0) - g_{13}([\omega; \mu]_{23})(X(\omega) - X(\omega_0)),
\end{aligned}$$

for some $\gamma \in [0,1]$. Rearranging and simplifying gives

$$\begin{aligned}
-\tfrac{1}{2}V_{22}([\omega;\gamma]_{23})(\omega-\omega_0) &- V_{23}([\omega;\gamma]_{23})(X(\omega)-X(\omega_0)) \\
&\le V_2(\omega_0,\omega_0,X(\omega_0)) + \frac{(X(\omega)-X(\omega_0))^2}{2(\omega-\omega_0)}V_{33}([\omega;\gamma]_{23}) \qquad \text{(A.6)} \\
&\le -\left[\tfrac{1}{2}V_{11}([\omega;\lambda]_1) + V_{12}([\omega;\mu]_{23}) + \tfrac{1}{2}V_{22}([\omega;\gamma]_{23})\right](\omega-\omega_0) \\
&\qquad - \left[V_{13}([\omega;\mu]_{23}) + V_{23}([\omega;\gamma]_{23})\right](X(\omega)-X(\omega_0)).
\end{aligned}$$

Since X is continuous at ω_0, as $\omega \nearrow \omega_0$, the terms bounding the expression in (A.6) converge to 0, and so the term in (A.6) must also converge to 0. But, for ω close to ω_0, $V_{33}([\omega;\gamma]_{23}) < 0$, and so that term is bounded away from 0 from below by $V_2(\omega_0,\omega_0,X(\omega_0)) > 0$, contradiction.

If $V_2(\omega_0,\omega_0,X(\omega_0)) < 0$, a similar contradiction is obtained using the same argument applied to a sequence $\omega \searrow \omega_0$. ■

Under Assumptions 1 and 2, Lemma B implies that $X(\omega) \neq X^{FB}(\omega)$ at an interior ω if X is continuous there, $X^{FB}(\omega) \in \text{int}(\mathcal{X})$ and $V_2(\omega,\omega,X(\omega)) \neq 0$.

Lemma C *Suppose Assumptions 1 and 2 hold. For each non-empty compact interval* $[\underline{\omega},\overline{\omega}] \subset \Omega$, $X([\underline{\omega},\overline{\omega}])$ *is bounded.*

Proof. The continuity of V and the Maximum Theorem imply that the first-best X^{FB} is a continuous function on Ω.

Suppose that X is unbounded on $[\underline{\omega},\overline{\omega}]$ and let $\omega_n \in [\underline{\omega},\overline{\omega}]$, $n = 1,2,...$ be a sequence such that $x_n = X(\omega_n) \to \infty$ (the case $x_n \to -\infty$ is handled analogously). We may assume that, by taking subsequences if necessary, the sequence ω_n converges to some $\omega_0 \in [\underline{\omega},\overline{\omega}]$. There is $N \in \mathbb{N}$ such that $X(\omega_n) > X^{FB}(\omega_0)$ for all $n \geq N$. Assumption 2 implies that $V(\omega_0,\omega_0,X(\omega_n)) \to -\infty$.

For any $K > 0$ and $\varepsilon > 0$ let $N_1 \in \mathbb{N}$ be such that $V(\omega_0,\omega_0,X(\omega_{N_1})) < -K-\varepsilon$. Since $X(\omega_n) \to \infty$ we can assume that $X(\omega_{N_1}) > \sup X^{FB}(\omega_n)$. The continuity of V implies that there is an $N_2 \in \mathbb{N}$, $N_2 > N_1$, such that $V(\omega_n,\omega_n,X(\omega_{N_1}))$ is within ε of $-K-\varepsilon$ for all $n \geq N_2$; hence we have $V(\omega_n,\omega_n,X(\omega_{N_1})) < -K$ for all $n \geq N_2$. Assumption 2 implies that for each ω_n, $V(\omega_n,\omega_n,x)$ is strictly decreasing in x if $x > X^{FB}(\omega_n)$. Hence, $V(\omega_n,\omega_n,X(\omega_n)) < -K$ for all n sufficiently large. Since this is true for arbitrary K, we have a contradiction to the incentive compatibility of X. ■

Lemma D *Suppose either that $\mathcal{X}$ is compact or that Assumptions 1 and 2 hold. If $\omega \to \omega_0$, then $V(\omega_0, \omega_0, X(\omega)) \to V(\omega_0, \omega_0, X(\omega_0))$.*

Proof. Fix a compact neighborhood N in Ω containing ω_0. By Lemma C or directly by the compactness of $\mathcal{X}$, $X(N)$ is bounded. Hence, V is uniformly continuous on $N^2 \times \mathrm{cl}(X(N))$, where $\mathrm{cl}(\cdot)$ denotes the closure.

Fix $\varepsilon > 0$. Uniform continuity implies that there is a $\delta_1 > 0$ with $\{\omega \in \Omega : |\omega - \omega_0| < \delta_1\} \subset N$ such that for all $x \in X(N)$,

$$|\omega - \omega_0| < \delta_1 \implies |V(\omega_0, \omega, x) - V(\omega_0, \omega_0, x)| < \varepsilon.$$

For these ω, incentive compatibility implies

$$V(\omega_0, \omega_0, X(\omega_0)) \geq V(\omega_0, \omega, X(\omega)) > V(\omega_0, \omega_0, X(\omega)) - \varepsilon. \tag{A.7}$$

On the other hand, there is a $\delta_2 > 0$ with $\{\omega \in \Omega; |\omega - \omega_0| < \delta_2\} \subset N$ such that for all $x \in X(N)$,

$$\begin{aligned} |\omega - \omega_0| < \delta_2 \implies & |V(\omega, \omega_0, x) - V(\omega_0, \omega_0, x)| < \varepsilon/2 \\ & \text{and} \quad |V(\omega, \omega, x) - V(\omega_0, \omega_0, x)| < \varepsilon/2. \end{aligned}$$

Hence, for these ω, incentive compatibility implies

$$\begin{aligned} V(\omega_0, \omega_0, X(\omega)) &> V(\omega, \omega, X(\omega)) - \frac{\varepsilon}{2} \\ &\geq V(\omega, \omega_0, X(\omega_0)) - \frac{\varepsilon}{2} > V(\omega_0, \omega_0, X(\omega_0)) - \varepsilon. \end{aligned} \tag{A.8}$$

Therefore, for $\omega \in \Omega$ with $|\omega - \omega_0| < \min(\delta_1, \delta_2)$, we have

$$V(\omega_0, \omega_0, X(\omega_0)) - \varepsilon < V(\omega_0, \omega_0, X(\omega)) < V(\omega_0, \omega_0, X(\omega_0)) + \varepsilon,$$

where the first inequality is from (A.8) and the second (A.7). Letting ε go to zero proves the lemma. ■

Lemma E *Suppose Assumption 1 holds and that either $\mathcal{X}$ is compact or Assumption 2 holds. If, for any $\omega \in \Omega$, $X(\omega) = X^{FB}(\omega)$, then X is continuous at ω.*

Proof. Choose $\omega_0 \in \Omega$ and fix a compact neighborhood N in Ω containing ω_0. Consider a sequence $\omega_n \to \omega_0$ in N. By the compactness of $\mathcal{X}$ or Lemma

C (for Assumption 2), the sequence $X(\omega_n)$ has a convergent subsequence that converges to an $\widehat{x} \in \mathrm{cl}(X(N))$. By Lemma D, on that subsequence,

$$V(\omega_0, \omega_0, X(\omega_n)) \to V(\omega_0, \omega_0, X(\omega_0)). \tag{A.9}$$

By Assumption 1, $V(\omega_0, \omega_0, \cdot)$ has a unique maximum at $x = X^{FB}(\omega_0)$. Equation (A.9) then implies $\widehat{x} = X^{FB}(\omega_0)$ when $X(\omega_0) = X^{FB}(\omega_0)$, and so X is continuous at ω_0. ■

Lemma F *Suppose Assumptions 1 and 2 hold. For any $\omega \in \Omega$, if $V_3(\omega, \omega, x) \neq 0$ for all $x \in \mathcal{X}$, then X is differentiable at ω.*

Proof. Choose $\omega_0 \in \Omega$ and derive $\widehat{x}$ and (A.9) as in the proof of Lemma E. Since $V(\omega, \omega, \cdot)$ is strictly monotone by assumption, it is one-to-one. Hence, $\widehat{x} = X(\omega_0)$, and X is continuous at ω_0. Differentiability then follows from Lemma A. ■

Lemma G *Suppose that Assumptions 1 and 2 hold, $V_{13}(\omega, \omega, x) \neq 0$ and $V_2(\omega, \omega, X(\omega)) \neq 0$ for all $(\omega, x) \in \mathrm{int}(\Omega) \times \mathcal{X}$. Then X can have at most one point of discontinuity ω_0 in $\mathrm{int}(\Omega)$. At the discontinuity,*

1. *X is continuous from either the left or the right,*
2. *the left-hand and the right-hand limits exist (i.e., the discontinuity is a jump discontinuity), and*
3. *the jump of X is of the same sign as V_{13}, i.e.,*

$$\left(\lim_{\omega \searrow \omega_0} X(\omega) - \lim_{\omega \nearrow \omega_0} X(\omega) \right) \cdot V_{13} > 0, \tag{A.10}$$

   and

4. *the jump is over X^{FB}, i.e.,*

$$\liminf_{\omega \to \omega_0} X(\omega) < X^{FB}(\omega_0) < \limsup_{\omega \to \omega_0} X(\omega). \tag{A.11}$$

Proof. Suppose X is discontinuous at some $\omega_0 \in \mathrm{int}(\Omega)$ and fix a compact neighborhood in Ω around ω_0, $[\omega_0 - \eta, \omega_0 + \eta] \cap \Omega$. By Lemma C, there exists a sequence $\{\omega_n\}$ in $[\omega_0 - \eta, \omega_0 + \eta] \cap \Omega$ with $\omega_n \to \omega_0$ such that the sequence $X(\omega_n)$ converges to some $\widehat{x} \neq X(\omega_0)$.

By the continuity of V and Lemma D,

$$V(\omega_0, \omega_0, \widehat{x}) = V(\omega_0, \omega_0, X(\omega_0)). \tag{A.12}$$

The strict quasi-concavity of V (from Assumptions 1 and 2) implies that the equation $V(\omega_0, \omega_0, x) = V(\omega_0, \omega_0, X(\omega_0))$ can have at most two distinct solutions in x, one of them being $X(\omega_0)$. If there is only one such solution, X is continuous at ω_0. Hence, suppose there are two and denote them by x' and x'', with $x' < x''$. Equation (A.12) and the strict quasi-concavity also implies that

$$x' < X^{FB}(\omega_0) < x''. \tag{A.13}$$

We now show that the left- and right-hand limits of X at ω_0 exist. First, consider any sequence $\omega_n \searrow \omega_0$, and let $x^+ = \lim_{\omega_n \searrow \omega_0} X(\omega_n)$. Focusing on the left-most and right-most terms of the inequality chain (A.5) and dividing through by $\omega - \omega_0$ yields, for $\omega = \omega_n$,

$$\left[\tfrac{1}{2} g_{11}([\omega_n; \lambda]_1) + g_{12}([\omega_n; \mu]_{23})\right](\omega_n - \omega_0) + g_{13}([\omega_n; \mu]_{23})(X(\omega_n) - X(\omega_0)) \geq 0. \tag{A.14}$$

Since $g_{13}([\omega; \mu]_{23}) = V_{13}([\omega; \mu]_{23})$, in the limit, this implies

$$(x^+ - X(\omega_0))V_{13}(\omega_0, \omega_0, \mu X(\omega_0) + (1 - \mu)x^+) \geq 0. \tag{A.15}$$

Now consider any sequence $\omega_n \nearrow \omega_0$, and let $x^- = \lim_{\omega_n \nearrow \omega_0} X(\omega_n)$. By the same argument as before,

$$(x^- - X(\omega_0))V_{13}(\omega_0, \omega_0, \mu X(\omega_0) + (1 - \mu)x^-) \leq 0. \tag{A.16}$$

Suppose that $V_{13} > 0$ (the case $V_{13} < 0$ is handled similarly, with the relevant inequalities reversed). Inequalities (A.15) and (A.16) imply

$$\liminf_{\omega_n \nearrow \omega_0} X(\omega_n) \leq \limsup_{\omega_n \nearrow \omega_0} X(\omega_n) \leq X(\omega_0) \leq \liminf_{\omega_n \searrow \omega_0} X(\omega_n) \leq \limsup_{\omega_n \searrow \omega_0} X(\omega_n). \tag{A.17}$$

By our earlier argument, each of the five terms in (A.17) is either equal to x' or x''. Hence, exactly one of the four inequalities in (A.17) is strict and X is continuous from the left or from the right.

We now argue that the discontinuity at ω_0 is a jump discontinuity. Suppose, en route to a contradiction, that the right-most inequality in (A.17) is strict (the left-most inequality is handled similarly). Hence, there are sequences $\omega_n \searrow \omega_0$ and $\widetilde{\omega}_n \searrow \omega_0$ with $X(\omega_0) = \lim X(\omega_n) < X^{FB}(\omega_0) < \lim X(\widetilde{\omega}_n)$. Since X^{FB} is continuous, for large n and small ε, we have $X(\widetilde{\omega}_n) > X^{FB}(\omega) > X(\omega_n)$ for all $\omega \in [\omega_0, \omega_0 + \varepsilon] \subset \mathrm{int}(\Omega)$. Hence, for $\omega \in [\omega_0, \omega_0 + \varepsilon]$, $X^{FB}(\omega)$ is not on the boundary of $\mathcal{X}$, and therefore $V_3(\omega, \omega, X^{FB}(\omega)) = 0$.

Fix ω_n and $\widetilde{\omega}_m$ with $\omega_0 < \widetilde{\omega}_m < \omega_n < \omega_0 + \varepsilon$. Lemmas B and E imply that for all $\omega \in [\widetilde{\omega}_m, \omega_n]$, $X(\omega) \neq X^{FB}(\omega)$. Hence, X is discontinuous at some $\widehat{\omega} \in [\widetilde{\omega}_m, \omega_n]$ with some left limit exceeding $X^{FB}(\widehat{\omega})$ and some right limit being less than $X^{FB}(\widehat{\omega})$. But this contradicts (A.17), applied to $\widehat{\omega}$. Hence, the right-hand limit at ω_0 exists. Similarly, one shows that the left-hand limit exists. Moreover, (A.17) implies (A.10), i.e., jumps can only be in one direction.

It remains to argue that X can have at most one discontinuity in $\text{int}(\Omega)$. Since all discontinuities are jump discontinuities, (A.13) implies that all discontinuities are isolated. Suppose there exist two discontinuities $\omega_1 < \omega_2$ such that X is continuous on (ω_1, ω_2). Because jumps can only be in one direction, (A.13) holds, and X^{FB} is continuous, it follows that there exists an $\omega \in (\omega_1, \omega_2)$ such that $X(\omega) = X^{FB}(\omega)$. By continuity, the set $E = \{\omega \in (\omega_1, \omega_2);\ X(\omega) = X^{FB}(\omega)\}$ is compact. Let $\widehat{\omega} \in (\omega_1, \omega_2)$ be its minimum. Since X is continuous at $\widehat{\omega}$ and $X(\widehat{\omega}) = X^{FB}(\widehat{\omega})$, Lemma B implies that $X^{FB}(\widehat{\omega})$ is on the boundary of $\mathcal{X}$. But this is impossible, since X is one-to-one and continuous on an open neighborhood containing $\widehat{\omega}$.

Finally, (A.11) is immediate from (A.13). ■

Lemma H *Suppose Assumptions 1 and 2 hold. Let Ω_0 be an open subset of Ω on which X is differentiable. Assume that $V_2(\omega, \omega, X(\omega)) \neq 0$ for all $\omega \in \Omega_0$. Then*

$$V_{12}(\omega, \omega, X(\omega)) + X'(\omega) V_{13}(\omega, \omega, X(\omega)) \geq 0 \tag{A.18}$$

for all $\omega \in \Omega_0$.

Proof. By Lemma A, $X'(\omega) \neq 0$ for all $\omega \in \Omega_0$, and so X^{-1} is differentiable for all $x \in X(\Omega_0)$. Since $X(\omega)$ must maximize $V(\omega, X^{-1}(x), x)$, the implied first order condition evaluated at $\omega = X^{-1}(x)$,

$$V_2(X^{-1}(x), X^{-1}(x), x)\frac{dX^{-1}(x)}{dx} + V_3(X^{-1}(x), X^{-1}(x), x) = 0,$$

must hold for all $x \in X(\Omega_0)$. Since the equation is an identity, the first derivative must also equal zero:

$$[V_{12} + V_{22}]\left(\frac{dX^{-1}}{dx}\right)^2 + [V_{13} + 2V_{23}]\frac{dX^{-1}}{dx} + V_2 \frac{d^2 X^{-1}}{dx^2} + V_{33} = 0, \tag{A.19}$$

where all the partial derivatives of V are evaluated at $(X^{-1}(x), X^{-1}(x), x)$.

The second derivative of $V(\omega, X^{-1}(x), x)$ is

$$\frac{d}{dx^2}V(\omega, X^{-1}(x), x) = V_{22}\left(\frac{dX^{-1}}{dx}\right)^2 + 2V_{23}\frac{dX^{-1}}{dx} + V_2\frac{d^2X^{-1}}{dx^2} + V_{33}, \quad \text{(A.20)}$$

where now all partial derivatives are evaluated at $(\omega, X^{-1}(x), x)$. Evaluating (A.20) at $\omega = X^{-1}(x)$, and substituting from (A.19), yields

$$\begin{aligned} \frac{d}{dx^2}V(X^{-1}(x), X^{-1}(x), x) &= -V_{12}\left(\frac{dX^{-1}}{dx}\right)^2 - V_{13}\frac{dX^{-1}}{dx} \\ &= -\left(\frac{dX^{-1}}{dx}\right)^2 (V_{12} + X'V_{13}). \end{aligned} \quad \text{(A.21)}$$

Since $V(\omega, X^{-1}(x), x)$ must have a local maximum at $x = X(\omega)$, the right-hand side of (A.21) must be (weakly) negative for all $\omega \in \Omega_0$, which yields (A.18). ∎

B Proof of Theorem 2

Statement 1 The result follows from Lemmas A and B once we have proved that X is continuous in the interior of Ω.

Suppose that X is discontinuous at $\omega_0 \in \text{int}(\Omega)$. Assume that $V_{13}(\omega, \omega, x) > 0$ (the other case is analogous). By Lemma G, X is continuous for all $\omega \neq \omega_0$ and so, by Lemma B, differentiable at all such $\omega \in \text{int}(\Omega)$ with derivative

$$X'(\omega) = -\frac{V_2(\omega, \omega, X(\omega))}{V_3(\omega, \omega, X(\omega))}.$$

Since by (A.13) and Lemma G, $X(\omega)$ is strictly smaller than the first-best for $\omega < \omega_0$ and strictly greater for $\omega > \omega_0$, we have $(\omega - \omega_0)V_3(\omega, \omega, X(\omega)) < 0$ for $\omega \neq \omega_0$, and so $(\omega - \omega_0)X'(\omega)V_2(\omega, \omega, X(\omega)) > 0$ for $\omega \neq \omega_0$. Since $V_2(\omega, \omega, X(\omega))$ does not change sign on $\text{int}(\Omega)$ by assumption, X' has one sign for $\omega < \omega_0$ and the other sign for $\omega > \omega_0$.

By assumption, $V_3(\omega, \widehat{\omega}, X(\widehat{\omega}))/V_2(\omega, \widehat{\omega}, X(\widehat{\omega}))$ is a strictly monotone function of $\omega \in \text{int}(\Omega)$, and so

$$X'(\omega)V_2(\omega, \widehat{\omega}, X(\widehat{\omega}))\frac{\partial}{\partial\omega}\left[\frac{V_3(\omega, \widehat{\omega}, X(\widehat{\omega}))}{V_2(\omega, \widehat{\omega}, X(\widehat{\omega}))}\right] < 0 \quad \text{(B.1)}$$

for ω either below or above ω_0.

Suppose it is the former (the latter is handled similarly). Choose arbitrary $\omega' < \omega'' < \omega_0$ in $\mathrm{int}(\Omega)$. Consider $V(\omega'', X^{-1}(x), x)$ as a function of x. By the differentiability of X and the Intermediate Value Theorem, there exists an $\overline{\omega} \in (\omega', \omega'')$ such that

$$\begin{aligned}
V(\omega'', \omega', X(\omega')) &= V(\omega'', \omega'', X(\omega'')) - \bigg\{ V_2(\omega'', \overline{\omega}, X(\overline{\omega})) \frac{dX^{-1}(X(\overline{\omega}))}{dx} \\
&\qquad + V_3(\omega'', \overline{\omega}, X(\overline{\omega})) \bigg\} (X(\omega'') - X(\omega')) \\
&= V(\omega'', \omega'', X(\omega'')) - \bigg\{ V_2(\omega'', \overline{\omega}, X(\overline{\omega})) \left(-\frac{V_3(\overline{\omega}, \overline{\omega}, X(\overline{\omega}))}{V_2(\overline{\omega}, \overline{\omega}, X(\overline{\omega}))} \right) \\
&\qquad + V_3(\omega'', \overline{\omega}, X(\overline{\omega})) \bigg\} (X(\omega'') - X(\omega')) \\
&= V(\omega'', \omega'', X(\omega'')) - \left\{ \frac{V_3(\omega'', \overline{\omega}, X(\overline{\omega}))}{V_2(\omega'', \overline{\omega}, X(\overline{\omega}))} - \frac{V_3(\overline{\omega}, \overline{\omega}, X(\overline{\omega}))}{V_2(\overline{\omega}, \overline{\omega}, X(\overline{\omega}))} \right\} \\
&\qquad \times V_2(\omega'', \overline{\omega}, X(\overline{\omega}))(X(\omega'') - X(\omega')) \\
&= V(\omega'', \omega'', X(\omega'')) - \int_{\overline{\omega}}^{\omega''} \frac{\partial}{\partial \omega} \left(\frac{V_3(\omega, \overline{\omega}, X(\overline{\omega}))}{V_2(\omega, \overline{\omega}, X(\overline{\omega}))} \right) d\omega \\
&\qquad \times V_2(\omega'', \overline{\omega}, X(\overline{\omega}))(X(\omega'') - X(\omega')) \\
&> V(\omega'', \omega'', X(\omega'')).
\end{aligned}$$

The strict inequality (which follows from (B.1) and $\overline{\omega} < \omega''$) contradicts incentive compatibility.

Statement 2 Assume that $V_2(\omega, \omega, X(\omega)) > 0$ for all $\omega \geq \omega_1$ (the other case is handled similarly). Suppose that X is discontinuous at some $\omega_0 > \omega_1$ (by Lemma G, there can be no other discontinuity). Denote $\Omega_0 = \{\omega \in \Omega; \omega > \omega_1, \omega \neq \omega_0\}$. By Lemma A, X is differentiable on Ω_0, and by Lemma H, it satisfies (A.18) there.

Since $V_2(\omega, \omega, X(\omega)) \neq 0$, by assumption, the continuity of X at ω_1 and Lemma A imply that $|X'(\omega)| \to \infty$ as $\omega \to \omega_1$. Hence, (A.18) implies that X' must have the same sign as V_{13} for $\omega \in \Omega_0$ sufficiently close to ω_1.

Suppose $V_{13} > 0$ (the case $V_{13} < 0$ is handled similarly). Near ω_1, X' is arbitrarily large and so for all $\omega \in (\omega_1, \omega_0)$, $X(\omega) > X^{FB}(\omega)$, while (A.10) and (A.11) imply $\lim_{\omega \nearrow \omega_0} X(\omega) < X^{FB}(\omega_0)$, the desired contradiction.

C Proof of Theorem 3

Choose $\omega_0 \in \Omega$ and derive $\widehat{x}$ and (A.9) as in the proof of Lemma E. Since $V(\omega, \omega, \cdot)$ is strictly monotone by assumption, it is one-to-one. Hence, $\widehat{x} = X(\omega_0)$, and X is continuous at ω_0. Differentiability then follows from Lemma A.

D Proof of Theorem 4

Statement 1 This is Lemma F.

Statement 2 Choose $\omega_0 \in \Omega$ and derive $\widehat{x}$ and (A.9) as in the proof of Lemma E. If $V(\omega_0, \omega_0, \cdot)$ is monotone, it is strictly monotone (and so one-to-one) because of Assumption 2. Hence, $\widehat{x} = X(\omega_0)$, and so X is continuous at ω_0. Differentiability therefore follows from Lemmas A and B.

Statement 3 By Lemma G, X can have at most one discontinuity in $\mathrm{int}(\Omega)$, say ω_0. By Lemma A, X is differentiable on $\mathrm{int}(\Omega) \diagdown \{\omega_0\}$, and by Lemma H, it satisfies (A.18) there.

Since $V_{12} \leq 0$ and $V_{13} \neq 0$, (A.18) immediately implies that X' must have the same sign as V_{13} for all $\omega \in \mathrm{int}(\Omega) \setminus \{\omega_0\}$.

Recall that for all $\omega \in \mathrm{int}(\Omega) \setminus \{\omega_0\}$,

$$\begin{aligned} X(\omega) > X^{FB}(\omega) &\Leftrightarrow V_3(\omega, \omega, X(\omega)) < 0 \\ &\Leftrightarrow X'(\omega) V_2(\omega, \omega, X(\omega)) > 0, \end{aligned}$$

where the first equivalence follows from the definition of the first-best and the second from the form of the derivative of X derived in Lemma A. By (A.13), we have both $X(\omega') > X^{FB}(\omega')$ and $X(\omega'') < X^{FB}(\omega'')$ for some $\omega', \omega'' \in \mathrm{int}(\Omega)$ (since ω_0 is a point of discontinuity). Since V_2 does not change sign on $\mathrm{int}(\Omega)$, X' does. But X' must have the same sign as V_{13} on $\mathrm{int}(\Omega) \diagdown \{\omega_0\}$, which does not change its sign by assumption. Contradiction.

E Proof of Theorem 5

This is Lemma E.

F Proof of Theorem 6

First note that it suffices to prove the result assuming Ω is open (since if Ω includes a boundary, and X is incentive compatible on the interior of Ω, then continuity implies X is incentive compatible on Ω).

1. (Sufficiency of global single crossing for IC) Since X satisfies (DE), it satisfies the first order condition implied by (IC), and so satisfies (IC) if

$$\frac{d}{dx}V(\omega, X^{-1}(x), x) \cdot (x - X(\omega)) \leq 0 \qquad \forall x \in X(\Omega), \omega \in \Omega. \tag{F.1}$$

The derivative equals

$$\begin{aligned}
&V_2(\omega, X^{-1}(x), x) \left(\left. \frac{dX}{d\widehat{\omega}} \right|_{X^{-1}(x)} \right)^{-1} + V_3(\omega, X^{-1}(x), x) \\
&\quad = V_2(\omega, X^{-1}(x), x) \frac{-V_3(X^{-1}(x), X^{-1}(x), x)}{V_2(X^{-1}(x), X^{-1}(x), x)} + V_3(\omega, X^{-1}(x), x) \\
&\quad = V_2(\omega, X^{-1}(x), x) \left\{ \frac{V_3(\omega, X^{-1}(x), x)}{V_2(\omega, X^{-1}(x), x)} - \frac{V_3(X^{-1}(x), X^{-1}(x), x)}{V_2(X^{-1}(x), X^{-1}(x), x)} \right\}.
\end{aligned}$$

If X is strictly increasing and $V_2(\omega, \omega, X(\omega)) > 0$ (the other possibilities handled mutatis mutandis), (F.1) is satisfied when

$$V_3(\omega, \widehat{\omega}, X(\widehat{\omega}))/V_2(\omega, \widehat{\omega}, X(\widehat{\omega}))$$

is an increasing function of ω for all $\widehat{\omega} \in \Omega$.

2. (Necessity of local single crossing for IC) Suppose X satisfies (IC). The second order condition is

$$\left. \frac{d^2}{dx^2} V(\omega, X^{-1}(x), x) \right|_{x=X(\omega)} \leq 0,$$

which is, after substituting for $d^2X^{-1}(x)/dx^2$ (see (A.21)),

$$-\frac{dX^{-1}}{dx} \cdot \left\{ V_{12}(\omega, \omega, X(\omega)) \frac{dX^{-1}}{dx} + V_{13}(\omega, \omega, X(\omega)) \right\} \leq 0,$$

i.e.,

$$-\frac{dX^{-1}}{dx} \cdot \left\{ V_{13}(\omega, \omega, X(\omega)) - V_{12}(\omega, \omega, X(\omega)) \frac{V_3(\omega, \omega, X(\omega))}{V_2(\omega, \omega, X(\omega))} \right\} \leq 0.$$

Multiplying both sides of this inequality by $-(X')^2$ yields an expression equivalent to (17), since

$$\frac{d}{d\omega}\left\{\frac{V_3(\omega,\widehat{\omega},X(\widehat{\omega}))}{V_2(\omega,\widehat{\omega},X(\widehat{\omega}))}\right\}\bigg|_{\widehat{\omega}=\omega} = \frac{1}{V_2(\omega,\omega,X(\omega))}\Big\{V_{13}(\omega,\omega,X(\omega)) - V_{12}(\omega,\omega,X(\omega))\frac{V_3(\omega,\omega,X(\omega))}{V_2(\omega,\omega,X(\omega))}\Big\}.$$

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