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Optimal Voting Rules

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Abstract

We study dominant strategy incentive compatible (DIC) and deterministic mechanisms in a social choice setting with several alternatives. The agents are privately informed about their preferences, and have single-crossing utility functions. Monetary transfers are not feasible. We use an equivalence between deterministic, DIC mechanisms and generalized median voter schemes to construct the constrained-efficient, optimal mechanism for an utilitarian planner. Optimal schemes for other welfare criteria such as, say, a Rawlsian maximin can be analogously obtained.

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1 Introduction

The use of the utilitarian principle as a guide for collective decision making or policy goes back to the birth of Economics. This principle uses cardinal information about preferences (and hence about preference intensities) and evaluates social outcomes in terms of the sum of agents' expected utilities, or, equivalently, in terms of average individual expected utility. When monetary transfers are feasible, the maximization of average utility is also a prerequisite for another classical desideratum, Pareto-efficiency.

In practice, however, many collective decisions are taken through simpler mechanisms that only extract ordinal information about the ranking of alternatives, and that do not allow monetary transfers among agents even if these would be feasible (e.g., various voting schemes within committees and legislatures). The lack of monetary transfers both makes it impossible to extract refined information about preference intensities, and also weakens the implications of Pareto efficiency: the set of Pareto efficient allocations can be very large, and it is not clear how to choose among efficient rules while only using the ordinal information obtained via voting.

Consider for example a simple situation with two outcomes, a *status-quo* and a *reform,* where voting by majority is known to have many desirable properties (including the provision of incentives for truthful voting). Should the society use simple majority voting even in a situation where it is a-priori known that a reform's proponent stands to gain a moderate amount if the reform is adopted, while a proponent of the status-quo may lose a lot in that case? Or should the society adopt the reform only if a sufficiently large majority (i.e., a *qualified-* or *supermajority*) votes in its favor?[1] It should be obvious that, in spite of the fact that any underlying qualified majority mechanism still uses only ordinal information about preferences and does not allow for monetary transfers, the selective application of such schemes - preferring the moderate gains from reform only if they accumulate to many agents - strongly points to an utilitarian reasoning.

The generalization of the above simple idea to settings with more than two alternatives is the main topic of the present paper: we derive the ex-ante welfare maximizing (e.g., utilitarian) mechanisms for settings with an arbitrary number of alternatives fully taking into account the strategic incentives that agents naturally face in such situations.

While the idea of comparing voting rules in terms of the ex-ante (cardinal) expected utility they generate is not new, most of the existing literature focused on settings with only two social alternatives, or, when more alternatives were considered, neglected the fact that agents may vote strategically (see the Literature Review below).

We study dominant strategy incentive compatible (DIC)[2] and deterministic mechanisms in a social choice setting where agents are privately informed and have single crossing utility

[1]Such "biased" rules are indeed often used when constitutional amendments are considered, e.g., both the German and the US constitutions can be changed only if two-thirds of the parliaments' members are in favor.

[2]Some authors use the term "strategy proof" mechanisms.

functions over several alternatives. We assume that monetary transfers are not feasible. This framework yields a social choice model where the set of DIC mechanisms is not trivial, while being markedly different from, say, the Vickrey-Clarke-Groves schemes that are associated with the DIC requirement in settings with monetary transfers and quasi-linear utility.

Our analysis uses a characterization result (Saporiti [2009]) for DIC, Pareto efficient and anonymous mechanisms in frameworks with single-crossing preferences. In turn, Saporiti's result builds on a classical "converse" to the *Median Voter Theorem* due to Moulin [1980]. Roughly speaking, Moulin's result says that, on the full domain of single-peaked preferences, all DIC, Pareto efficient and anonymous mechanisms that only depend on the agents' top alternatives (or peaks) can be described as schemes that choose the median among the n "real" peaks of actual voters and an additional $n-1$ fixed "phantom" voters' peaks.[3] Saporiti is able to remove Moulin's assumption that the allowed mechanisms only depend on peaks, while obtaining a result in the same spirit as Moulin's which holds for maximal domains of single-crossing preferences.

Although deterministic, DIC mechanisms are described here by a function of several continuous variables satisfying complex constraints, the above result implies that the problem of finding mechanisms maximizing some given social welfare functional (that may depend on preference intensities!) can be reduced to the simpler problem of finding K non-negative constants adding up to $n-1$, where K is the number of alternatives and n is the number of agents. These constants represent the number of phantom peaks that need to be placed on each alternative.

For example, when there are only two alternatives, say a *status quo* and a *reform*, locating m, $0 \leq m \leq n-1$, phantom peaks on the reform and $n-1-m$ phantom peaks on the status-quo yields a choice rule where the reform is implemented if at least $n-m$ real voters are in its favor. The optimal m in this case, and, more generally, the optimal numbers of phantom peaks on each of alternative, depend of course on the primitives of the social choice situation such as the utility functions and the distribution of (real) agents' types.

Our main result focuses on the utilitarian maximization of social welfare. Under a joint regularity assumption on utilities and on the distribution of types, we offer simple and intuitive formulae for the optimal number of phantom peaks that need to be placed on each alternative. The formulae are obtained by observing that shifting phantom peaks among adjacent alternatives has an effect only in cases where the median peak shifts as well. Optimal schemes for other welfare criteria such as, say, a Rawlsian maximin, or maximax can be analogously obtained.

We also show that, although the first-best utilitarian rule is **not** implementable in our setting, the second-best (constrained efficient) rule obtained here approximates the first-best if the population is large. In other words, with large populations, the optimal placement of phantoms is such that the decision is shifted from the alternative preferred by the (true)

[3]The precise result requires several technical conditions.

median voter in the direction of the alternative preferred by the average voter.

A very well known application of a social choice framework with single-crossing preferences is to voting over tax schedules - see for example, Roberts [1977], Romer [1975], and Meltzer and Richard [1981]. Persson and Tabellini [2000] (Chapter 6) survey this literature and use such a framework as the starting point to study voting for redistributive programs such as pensions, unemployment insurance, assistance to the poor, and labor market regulations. They assume that there are only two parties who each suggests policy platforms. Hence, although the taxation application can have infinite number of alternatives, it reduces there to the case of two alternatives. Under simple majority voting, the decisive voter is the one with the median type. Given that the real income distribution typically has a mean above median, the government "size" that results from majority voting in this model is too large compared to the social optimal one. This basic insight is also important in understanding the difficulty of implementing pension reforms. Our framework may thus be also useful for the analysis of multi-party elections. As we shall see later, for any number of alternatives, our optimal mechanism tailors the voting rule to correct the difference between the mean and the median.

Related Literature

The present analysis combines modeling choices and insights from two important strands of the literature:

1. On the one hand, the private values model with monetary transfers serves as the workhorse of a very large body of literature that focuses on trading mechanisms for the provision of public or private goods. Classical results include, for example, the characterization of utilitarian (value-maximizing) mechanisms due to Vickrey-Clarke-Groves, and the characterization of revenue-maximizing auctions due to Myerson [1981]. Cardinal preference intensities play a main role in the formulation of both implementability and optimality results. In addition, monetary transfers are key to controlling the agents' incentives, and can be finely tuned to match the values obtained from physical allocations. A main complication in our present framework without transfers (but otherwise similar to the above) is that implementability puts restrictions on mechanisms that do not reduce to a simple monotonicity condition.

2. On the other hand, a distinct, very large body of work in the realm of social choice has focused on the implementation of desirable social choice rules in abstract frameworks with purely ordinal preferences, and without monetary transfers. Classical results include the Gibbard-Satterthwaite *Impossibility Theorem* (Gibbard [1973] and Satterthwaite [1975]) and the *Median Voter Theorem* for settings with single-peaked preferences (see Black [1948]). When a Pareto-efficient rule, say, is not implementable in a certain framework, that literature often remains silent about how to choose among

implementable schemes because preference intensities are not part of the model, and because other goals are not easily formulated within it. For similar reasons, when multiple Pareto-efficient rules are implementable, this literature does not offer tools to meaningfully ranking them.

The idea of comparing voting rules in terms of the ex-ante expected utility they generate goes back to Rae [1969]. That paper and almost the entire following literature focus on settings with two social alternatives (a reform and a status quo, say) where a mechanism can be described by a single function, the probability that the reform is chosen given the agents' reports about their types. In this special case, the DIC constraint implies that deterministic mechanisms are, for any profile of others' reports, described by a step function with a unique jump. As a consequence of this simple structure, anonymous and constrained-efficient mechanisms can be represented by qualified majority rules where the reform is chosen if at least a certain number of agents votes in its favor. Schmitz and Tröger [2012] identify qualified majority rules as ex-ante welfare maximizing in the class of DIC mechanisms - as explained above this can be seen as an implication of our main result.[4] Azrieli and Kim [2011] nicely complement this analysis for two alternatives by showing that any interim Pareto efficient, Bayesian incentive compatible (BIC) choice rule must be a qualified majority rule.[5] The situation dramatically changes when there are three, or more alternatives: the DIC/BIC constraints and the mechanisms themselves are much more complex, and not much is known about them.

Apesteguia, Ballester and Ferrer [2011] consider a general social choice model where agents derive cardinal utility from several alternatives, and evaluate mechanisms in terms of the ex-ante expected utility they generate.[6] Their analysis completely abstracts from incentives constraints: strategic voting is not considered - this would lead there to impossibility results- and the scoring rules that emerge as optimal in their analysis are known to be subject to strategic manipulation.

Borgers and Postl [2009] study a setting with three alternatives: in their model it is common knowledge that the top alternative for one agent is the bottom for the other, and vice-versa. The agents also differ in the relative intensity of their preferences for a middle alternative (the *compromise*) when compared to the top and bottom one, respectively. This intensity is private information. In addition to a characterization of BIC mechanisms in terms of monotonicity and an envelope condition, Borgers and Postl conduct numerical simulations and show that the efficiency loss from second-best rules is often small.[7]

[4] These authors also perform an analysis for Bayesian mechanisms, which is not covered by our study.

[5] Again in a setting with two alternatives, Barbera and Jackson [2006] take the qualified majority rule as given, and derive the optimal weight that maximizes the total expected utilities of all agents.

[6] They also consider other goals such as maxmin, etc.

[7] See also McLean and Postlewaite [2002] who study Bayesian incentive compatibility in settings where monetary transfers are limited.

In a principal-agent model with quadratic utility functions, hidden information but without transfers, Kovac and Mylovanov [2009] find that the optimal mechanism is deterministic. Rosar [2012] looks at a setting with a continuum of alternatives, quadratic utilities and interdependent preferences, and compares two particular aggregation mechanisms, the median and the average, respectively.

Motivated by computer science applications, Hartline and Roughgarden [2008] study how the system designer can use service degradation (money burning) to align the private users' interests with the social objective. Chakravarty and Kaplan [2013] and Condorelli [2012] analyze optimal allocation problems in private value environments without monetary transfers. In their models agents can send costly and socially wasteful signals (these may be payments to outsiders). In contrast to the above papers, Drexel and Klein [2013] allow the redistribution of the collected monetary payments among the agents. They confine attention to settings with two social alternatives and show that a principal who wishes to maximize the agents' welfare (i.e., welfare from the physical allocation minus potential transfers to outsiders) will use a mechanism that does not involve any monetary transfers! Hence, their paper offers a powerful argument for the use of voting schemes. In particular, it must be the case that, for settings with two alternatives, their optimal mechanism coincides with the one derived in this paper, where monetary transfers are a-priori ruled out.

A quite different line of study is pursued by Jackson and Sonnenschein [2007] who consider the linkage of many distinct social problems. Even if no monetary transfers within one problem are possible, the linkage with other decisions creates the possibility of fine-tuning incentives, which acts as having some "pseudo-transfers". Efficiency can be attained then in the limit, where the number of considered problems grows without bound.

As already mentioned above, the seminal paper in the social choice literature closely related to the present research is Moulin [1980]. Several authors have extended Moulin's characterization in terms of median choices and phantom voters within the single-peaked preferences domain by discarding the assumption that mechanisms can only depend on peaks.[8] Excellent examples are Barbera and Jackson [1994], Sprumont [1991], Ching [1997], Schummer and Vohra [2007], and Chatterjee and Sen [2011].[9] Saporiti [2009] extends Moulin's characterization to the domain of single-crossing preferences. On maximal single-crossing domains he obtains an equivalence between dominant strategy incentive compatible mechanisms and generalized median voter schemes. We will use his equivalence result to characterize optimal mechanisms.

[8]See Sprumont [1995] for an excellent survey. Recently, Ehlers, Peters and Storcken [2002] extend Moulin's characterization to probabilistic strategy-proof rules, and Nehring and Puppe [2007] extend it to a class of generalized single-peaked preference domains based on abstract betweenness relations.

[9]Schummer and Vohra [2002] and Dokow et al. [2012] also establish the equivalence between strategy-proof rules and generalized median voter schemes. In both models, however, agents' preferences are quadratic and thus parameterized solely by their peaks. Hence, they can directly focus on peaks-only mechanisms.

The rest of the paper is organized as follows: In Section 2 we describe the social choice model and the design problem. In Section 3 we illustrate the model and some implications of incentive compatibility in the simple special case where utilities are linear. In particular, we show that the welfare-maximizing rule (first-best) is not implementable although it is monotone. In Section 4 we present Saporiti's result that characterizes anonymous, Pareto-efficient and DIC mechanisms in terms of median mechanisms among the n real voters and $n-1$ phantom ones. We also show that, in the presence of DIC, Pareto-efficiency, unanimity and ontoness of the social choice function are all equivalent. In Section 5 we use the above characterization result to derive precise formulae governing the location of phantom peaks in the incentive compatible, welfare maximizing mechanism (second-best). We also show that, when the number of agents goes to infinity, the second best mechanism converges to the first-best that can be achieved under complete information. Finally, we also discuss extensions to other welfare criteria. Section 6 concludes. All proofs are in Appendix A. Appendix B discusses in more detail the regularity conditions used in the characterization of optimal mechanisms.

2 The Social Choice Model

We consider n agents who have to choose one out of K mutually exclusive alternatives. Let $\mathcal{K} = \{1, ..., K\}$ denote the set of alternatives. Agent $i \in \{1, ..., n\}$ has (cardinal) utility $u^k(x_i)$, where $k \in \mathcal{K}$ is the chosen alternative and where x_i is a parameter (or type) privately known to agent i only. We assume that for any k, $u^k(x_i)$ is bounded. The types $x_1, ..., x_n$ are distributed on the interval $[0, 1]^n$ according to a commonly known, joint cumulative distribution function Φ with density ϕ.[10]

Given any two alternatives k and l with $k < l$, let $x^{k,l}$ denote the cutoff type that is indifferent between them:[11]

$$u^l\left(x^{k,l}\right) = u^k\left(x^{k,l}\right). \tag{1}$$

To simplify notation, we denote $x^k \equiv x^{k-1,k}$. We assume that utilities are **single-crossing** with respect to the order of alternatives $1, ..., K$: for any two alternatives k and l with $k < l$ it holds that

$$\begin{cases} u^k\left(x_i\right) > u^l\left(x_i\right) & \text{if} \quad x_i < x^{k,l} \\ u^k\left(x_i\right) < u^l\left(x_i\right) & \text{if} \quad x_i > x^{k,l} \end{cases}. \tag{2}$$

We further assume that each alternative is the top alternative for some type of the agents. That is, for any $k \in \mathcal{K}$, there exists $x_i \in [0, 1]$ such that

$$u^k\left(x_i\right) > \max_{l \in \mathcal{K}, l \neq k} u^l\left(x_i\right). \tag{3}$$

[10]Here agents' types can be correlated. In Section 5 we shall assume independence between the agents' types.

[11]We assume that for any two alternatives k and l, the indifferent type $x^{k,l}$ is unique.

Note that we use here the one-dimensional, private values specification – the most common one in the vast literature on optimal mechanism design with monetary transfers that followed Myerson's (1981) seminal contribution. But we assume that monetary transfers are not feasible in our framework.

The social planner's general goal is to reach, for any realization of types, a Pareto efficient allocation. We shall primarily focus on the case of a utilitarian planner whose objective is to maximize the sum of the agents' expected utilities

$$\max_{k\in\mathcal{K}} E\left[\sum_i u^k\left(x_i\right)\right].$$

Remark 1 *The single-crossing property (2), together with assumption (3), implies that the cutoffs are well-ordered:*

$$0 \equiv x^1 < ... < x^K < x^{K+1} \equiv 1. \tag{4}$$

To see this, we note that, by definition of x^k and the single-crossing property (2),

$$u^k\left(x_i\right) < u^{k-1}\left(x_i\right) \text{ for all } x_i < x^k.$$

Similarly, by definition of x^{k+1} and the single-crossing property (2), we have

$$u^k\left(x_i\right) < u^{k+1}\left(x_i\right) \text{ for all } x_i > x^{k+1}.$$

If $x^k \geq x^{k+1}$, any type x_i satisfies either $x_i \leq x^k$ or $x_i \geq x^{k+1}$, and thus alternative k is (weakly) dominated either by alternative $k-1$ or by alternative $k+1$, which contradicts (3). Therefore, we must have $x^k < x^{k+1}$ for all $k \in \mathcal{K}$, which proves (4). Note that, by definition of x^k and by (4), agents with type x_i have k as their top alternative if and only if $x_i \in \left[x^k, x^{k+1}\right]$.

Remark 2 *The underlying domain of ordinal preferences is **maximal** with respect to single-crossing, i.e., one cannot add to it another ordinal preference profile without violating the single crossing property. To see this, note that each cutoff $x^{k,l}$ divides the set of types into two intervals where ordinal preferences differ with respect to the ordering of alternative k and l. Since each alternative is top for some types, the interval of types is thus partitioned into $K(K-1)/2+1$ parts, each corresponding to a distinct ordinal preference. But this is also the maximum number of ordinal profiles in a maximal domain of single-crossing preferences on K alternatives (see Saporiti [2009]).*

Remark 3 *The agents' preferences are single-peaked. To see this, consider agent i with type $x_i \in \left(x^k, x^{k+1}\right)$. By definition of x^k, agent i prefers alternative k to any alternative $l < k$, and by definition of x^{k+1}, agent i prefers k over any $l > k$. Now consider two alternatives l and m with $l < m < k$. Since $x^l < x^m < x^k$, we have $x_i > x^{l,m}$ and agent i prefers m to l. Similarly, agent i prefers m to l if $k < m < l$. Therefore, agent i's preferences are single-peaked. On the other hand, our preference domain is a **strict subset** of the full single-peaked preference domain: not all single-peaked preferences are compatible with our environment (see below an explicit illustration in the linear environment).*

We focus on deterministic direct mechanisms where each agent reports his type, and where, for any profile of reports, the mechanism chooses one alternative from $\mathcal{K}$. Formally, a deterministic direct mechanism is a function $g : [0,1]^n \to \mathcal{K} = \{1, ..., K\}$. A deterministic mechanism is *dominant strategy incentive compatible* (DIC) if for any player i and for any x_i, x_i' and x_{-i}:

$$u^{g(x_i,x_{-i})}(x_i) \geq u^{g\left(x_i',x_{-i}\right)}(x_i). \tag{5}$$

It is clear from the above definition that two types that have the same ordinal preferences must be treated in the same way by a DIC mechanism. Thus, an implication of the lack of monetary transfers is that DIC mechanisms cannot depend on preferences intensities.

3 An Illustration: The Linear Environment

We introduce here the following simple specification with linear utilities - these are necessarily single crossing. This specification will also be very useful later, when we describe optimal mechanisms.

Suppose the utilities are linear: $u^k(x_i) = a_k + b_k x_i$. We also assume that $b_k \geq 0$ for all $k \in \mathcal{K}$ and $b_k \neq b_l$ for all $l \neq k$. Without loss (by renaming alternatives if necessary), we assume that $b_K > b_{K-1} > ... > b_1 \geq 0$. The cutoff type who is indifferent between two adjacent alternatives k and $k-1$ is given by

$$x^k \equiv x^{k-1,k} = \frac{a_{k-1} - a_k}{b_k - b_{k-1}}. \tag{6}$$

These restrictions, together with the definition of $x^{l,k}$, imply that $x^{l,k} \in (x^{l+1}, x^k)$ for $k > l+1$, because

$$x^{l,k} = \frac{a_l - a_k}{b_k - b_l} = \frac{(a_l - a_{l+1}) + ... + (a_{k-1} - a_k)}{(b_{l+1} - b_l) + ... + (b_k - b_{k-1})}.$$

Note first that our preference domain is a strict subset of the full single-peaked preference domain. Indeed, consider a setting with 4 different alternatives $(1, 2, 3$ and $4)$, and suppose that it holds that $x^{1,4} \in \left(x^{2,3}, x^{3,4}\right)$, as shown in Figure 1. The feasible single-peaked preferences that have alternative 2 on their top are $2 \succ 1 \succ 3 \succ 4$ and $2 \succ 3 \succ 1 \succ 4$. In particular, the preference $2 \succ 3 \succ 4 \succ 1$ is not compatible with the linear environment. Similarly, if $x^{1,4} \in \left(x^{1,2}, x^{2,3}\right)$, the feasible single-peaked preferences that have alternative 3 on their top are $3 \succ 2 \succ 4 \succ 1$ and $3 \succ 4 \succ 2 \succ 1$. Here the preference profile $3 \succ 2 \succ 1 \succ 4$ is not compatible with our structure.

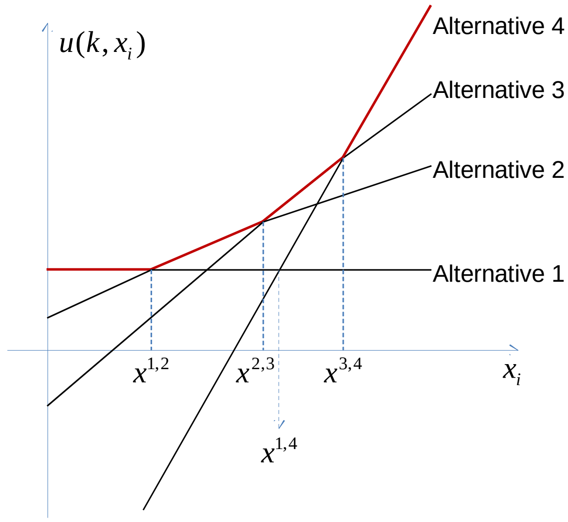

Figure 1: Not all single-peaked preferences are compatible with our linear structure.

Analogously to the classical framework with monetary transfers, a mechanism $g(x_i, x_{-i})$ is DIC if and only if (i) for all x_{-i} and for all i, $g(x_i, x_{-i})$ is increasing in x_i; and (ii) for any agent i, any $x_i \in [0, 1]$ and $x_{-i} \in [0, 1]^{n-1}$, the following condition holds:

$$u^{g(x_i, x_{-i})}(x_i) = u^{g(0, x_{-i})}(0) + \int_0^{x_i} b_{g(z, x_{-i})} dz. \tag{7}$$

When monetary transfers are feasible, **any** monotone decision rule $g(x_i, x_{-i})$ is incentive compatible since it is always possible to augment it with a transfer such that the equality required by (7) holds. Thus, with transfers, only monotonicity really matters for DIC. In particular, if monetary transfers were available, the welfare-maximizing allocation would be implementable via the well-known Vickrey-Clarke-Groves mechanisms. But, without monetary transfers not all monotone decision rules $g(x_i, x_{-i})$ are implementable, and in particular, the welfare maximizing allocation need not be incentive compatible although it is monotone. This phenomenon is illustrated in the next example.

Example 1 (First-best Rule Not Implementable) *Consider the linear environment with two alternatives $\{1, 2\}$ and with two agents $\{i, -i\}$. The designer is indifferent between alternatives 1 and 2 if*

$$2a_1 + b_1(x_i + x_{-i}) = 2a_2 + b_2(x_i + x_{-i}).$$

The first-best rule conditions on the value of the average type, and is given by

$$g(x_i, x_{-i}) = \begin{cases} 1 & \text{if } \frac{1}{2}(x_i + x_{-i}) \in [0, x^2) \\ 2 & \text{if } \frac{1}{2}(x_i + x_{-i}) \in [x^2, 1] \end{cases}$$

where cutoff x^2 is defined in (6): $x^2 \equiv (a_1 - a_2)/(b_2 - b_1)$. The first-best rule is increasing in both x_i and x_{-i}. But, for all $x_{-i} \in [0, 2x^2)$ and $x_i \in [2x^2 - x_{-i}, 1)$, we can rewrite the *integral condition (7) as*

$$a_2 + b_2 x_i = a_1 + \int_0^{2x^2 - x_{-i}} b_1 dz + \int_{2x^2 - x_{-i}}^{x_i} b_2 dz = a_1 + b_1 \left(2x^2 - x_{-i}\right) + b_2 \left(x_i - 2x^2 + x_{-i}\right),$$

which reduces to $x_{-i} = x^2$. Therefore, the integral condition is violated for all $x_{-i} \neq x^2$.

The following taxation example illustrates how the above linear specification can be used to model important voting applications. The example is taken from Persson and Tabellini [2000] (Chapter 6, page 118-121), which, in turn, follow Romer [1975], Roberts [1977], and Meltzer and Richard [1981].

Example 2 *Consider a static economy with a large number of agents who differ only in their productivities. The preferences of agent i are quasi-linear:*

$$u_i = c_i + v(\ell_i),$$

where c_i and ℓ_i denote consumption and leisure, respectively. The function $v(\cdot)$ is increasing and strictly concave. Each individual faces a private budget constraint:

$$c_i \leq k y_i + \tau,$$

where $(1-k)$ is the (linear) income tax rate, y_i the labor supply of agent i, and τ a lump-sum transfer. The real wage is normalized to one. Each individual also faces an "effective time" constraint:

$$y_i + \ell_i \leq 1 + x_i,$$

where x_i captures individual productivity for agent i. More productive individuals have a larger effective time x_i. Let's assume that x_i is distributed according to a common distribution F with mean μ.

Calculating the optimal labor supply, together with redistribution of the raised taxes through τ, allows us to write the indirect utility function of individual i over k as

$$u_i^k(x_i) = k\left(1 + x_i - v_\ell^{-1}(k)\right) + (1-k)(1 + \mu - \xi(k)) + v\left(v_\ell^{-1}(k)\right),$$

where $v_\ell^{-1}(\cdot)$ is the inverse function of the marginal utility of leisure. Taking into account the optimality of labor choice, the marginal utility with respect to k is:

$$\frac{du_i^k(x_i)}{dk} = x_i - \mu - (1-k)\frac{1}{v_{\ell\ell}\left[v_\ell^{-1}(k)\right]}. \tag{8}$$

It implies that the most preferred tax rate $(1-k_i^*)$ for agent i is

$$1 - k_i^* = v_{\ell\ell}\left[v_\ell^{-1}(k_i^*)\right](x_i - \mu).$$

Therefore, an agent with below average income $(x_i < \mu)$ prefers a positive tax $(1 - k_i^* > 0)$, and the preferred tax rate $(1-k_i^*)$ is higher the poor he is. We can translate this formulation into the linear specification by setting $b_k = k$ and

$$a_k = k\left(1 - v_\ell^{-1}(k)\right) + (1-k)\left(1 + \mu - v_\ell^{-1}(k)\right) + v\left(v_\ell^{-1}(k)\right).$$

4 Generalized Median Voter Schemes

In order to study optimal mechanisms, we first need to characterize the set of incentive compatible mechanisms. An influential paper by Moulin [1980] shows that, if each agent is **restricted** to report his top alternative only, then every DIC, Pareto efficient and anonymous voting scheme on the **full** domain of single-peaked preferences is equivalent to a generalized median voter scheme. That is, one can obtain each DIC, efficient and anonymous scheme by adding $(n-1)$ fixed ballots (phantoms) to the n voters' ballots and then choosing the median of this larger set of ballots. It turns out that Moulin's characterization also holds in our setting although agents are not restricted to report only their peaks, and although the domain of preferences is a strict subset of the full domain of single-peaked preferences. This result is due to Saporiti [2009]: he provides a characterization of anonymous, unanimous and DIC mechanisms for **maximal** domains of single-crossing preferences, in a spirit similar to Moulin [1980]. We first need several definitions:

Definition 1 *(i) A mechanism g is onto if, for every alternative $k \in \mathcal{K}$, there exists a type profile $(x_i, x_{-i}) \in [0,1]^n$ such that $g(x_i, x_{-i}) = k$.*

(ii) A mechanism g is unanimous if $x_i \in \left(x^k, x^{k+1}\right)$ for all i implies $g(x) = k$.

(iii) A mechanism g is Pareto efficient if, for any profile of reports $(x_i, x_{-i}) \in [0,1]^n$, there is no alternative $k \in \mathcal{K}$ such that $u_i(x_i, k) \geq u_i(x_i, g(x))$ for all i, with strict inequality for at least one agent.

(iv) A mechanism g is anonymous if, for any profile of reports $(x_i, x_{-i}) \in [0,1]^n$, $g(x_1, ..., x_n) = g\left(x_{\sigma(1)}, ..., x_{\sigma(n)}\right)$ where σ denotes any permutation of the set $\{1, ..., n\}$.

Theorem 1 (Saporiti, 2009) *An unanimous, anonymous mechanism g is DIC if and only if there exists $(n-1)$ numbers $\alpha_1, ..., \alpha_{n-1} \in \mathcal{K}$ such that for any type profile $(x_1, ..., x_n) \in [0,1]^n$ with $x_i \in (x^{k_i}, x^{k_i+1}]$ for all i, it holds that*

$$g(x_1, ..., x_n) = M(\alpha_1, ..., \alpha_{n-1}, k_1, ..., k_n),$$

where the function $M(\alpha_1, ..., \alpha_{n-1}, k_1, ..., k_n)$ returns the median of $(\alpha_1, ..., \alpha_{n-1}, k_1, ..., k_n)$.

It is clear that in our setting a Pareto-efficient mechanism is unanimous and that an unanimous mechanism is onto. In the presence of the DIC all these requirements are, in fact, equivalent.

Lemma 1 *Every onto and DIC mechanism is Pareto efficient and satisfies unanimity.*

Example 3 (Voting with Two Agents) *If $N = 2$ then all DIC, anonymous and unanimous mechanisms can be described by median mechanisms with one phantom. For example,* *if $K = 3$, the social alternatives chosen by all such mechanisms as a function of the agents' types can be described by the three tables in Figure 2 below.*

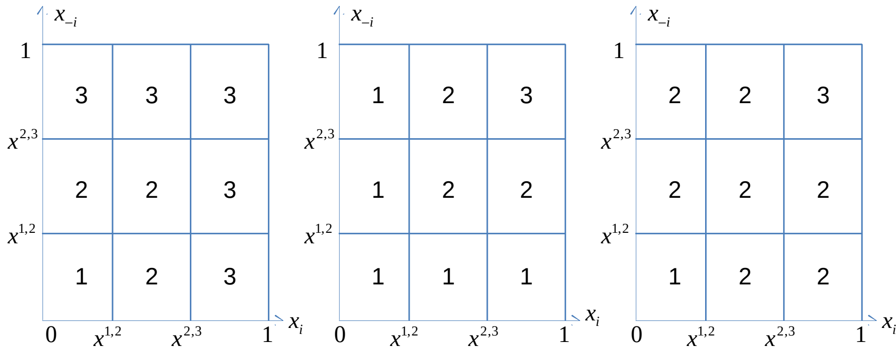

Figure 2: DIC mechanisms with two agents and three alternatives

The left table in Figure 2 corresponds to the case where the added phantom voter has a peak on $k^* = 3$. The middle table corresponds to the case where $k^* = 1$, while the right table corresponds to the case with $k^* = 2$.

As illustrated in the above Example, Theorem 1 greatly simplifies the problem of finding optimal mechanisms. Nevertheless, it should be clear that, with many agents and alternatives, this remains a rather complex discrete optimization problem since the number of possible partitions of phantoms' peaks is quite large.[12] In the next section we show how to solve it under some regularity conditions on the distribution of types and utility functions.

5 Optimal Mechanisms

In this section we characterize socially optimal allocations that respect the incentive constraints (constrained efficiency, or "second-best"). Following the mechanism design literature, we shall primarily focus on the utilitarian welfare criterion: the social planner wants to maximize the sum of the agents' expected utilities. As we mentioned previously, for DIC mechanisms the three properties used above – onto, Pareto efficiency, and unanimity – are equivalent in our environment. Therefore, we confine attention below to mechanisms that are anonymous, onto and DIC.

We first characterize the optimal mechanism for single-crossing preferences. We then use the linear case to further illustrate some of the properties of the optimal mechanism. Finally, we briefly discuss how to obtain similar characterizations using welfare criteria other than utilitarian.

5.1 General Characterization

Given the characterization of Theorem 1, the set of anonymous, onto and DIC mechanisms coincides with the set of generalized median voter schemes with n real voters and $(n-1)$

[12]The number of possible partitions can be calculated from the so called *Stirling numbers of the second kind.*

phantom voters. Therefore, the task of searching optimal mechanisms is reduced to that of finding the optimal position for the peaks of these $(n-1)$ phantom voters. In order to be able to offer simple, intuitive formulae for the number of phantom peaks on each alternative we make below several standard assumptions on the distribution of agents' signals.

Assumption A *The agents' signals are distributed identically and independently of each other on the interval* $[0,1]$ *according to a cumulative distribution function* F *with density* f.

This assumption yields the standard symmetric, independent private values model (SIPV) widely used in the literature on trading mechanisms with transfers. We need another assumption that combines requirements on the utility functions and on the distribution of types. Before introducing it, we need some notation. Let us define, for all $k \geq 2$,

$$u^k_{x<x^k} = E\left[u^k(x)\,|x < x^k\right]$$

as the expected utility from alternative k, conditional on the agent's type x being lower than the cutoff x^k. Similarly, we define

$$u^k_{x>x^k} = E\left[u^k(x)\,|x > x^k\right]$$

as the expected utility from alternative k conditional on the agent's type x being higher than the cutoff x^k. Finally, let us define the function $\Gamma(k)$ as

$$\Gamma(k) = \frac{u^{k-1}_{x<x^k} - u^k_{x<x^k}}{u^k_{x>x^k} - u^{k-1}_{x>x^k}}.$$

That is, $\Gamma(k)$ is the ratio of the difference of the "lower" conditional expected utilities for two adjacent alternatives $k-1$ and k over the difference of the "upper" conditional expected utilities for the same adjacent alternatives. By the definition of cutoff x^k and by the single-crossing property, $u^{k-1}_{x<x^k} > u^k_{x<x^k}$ and $u^k_{x>x^k} > u^{k-1}_{x>x^k}$. Therefore, $\Gamma(k) > 0$ for all $k \geq 2$. The function $\Gamma(k)$ plays an important role in our later analysis.

Assumption B *The function* Γ *is increasing.*

In Appendix B we derive an intuitive and concise sufficient condition on the primitives of the social choice model (utility functions and the distribution of types) for the above assumption to hold. Define also

$$\gamma(k) = \frac{\left(u^{k-1}_{x<x^k} - u^k_{x<x^k}\right)}{\left(u^{k-1}_{x<x^k} - u^k_{x<x^k}\right) + \left(u^k_{x>x^k} - u^{k-1}_{x>x^k}\right)} = \frac{1}{1 + 1/\Gamma(k)},$$

and note that, by Assumption B, γ is also increasing.

Consider now an environment with n voters, and let l_k denote the number of phantom voters with peak on alternative k in a generalized median voter scheme with $n-1$ phantom voters. Our analysis is based on a simple observation: if l_k is part of the optimal allocation of the $(n-1)$ phantoms and $l_k > 0$, then shifting one phantom voter from alternative k to either alternative $k-1$ or $k+1$ weakly reduces the total expected utility.[13] For instance, shifting one phantom voter from alternative k to $k-1$ has an impact only if it changes the chosen alternative. However, the shift will change the chosen alternative only if there are $(n-1)$ voters (both "real" and "phantom") with values below x^k, or, in other words, only if there are exactly $\left(n-1-\sum_{m=1}^{k-1} l_m\right)$ real voters with values below x^k. These kind of arguments generate the following bounds on the cumulative distribution of phantom voters:

$$\sum_{m=1}^{k-1} l_m \geq n\gamma(k) - 1, \text{ for all } k \geq 2, \tag{9}$$

$$\sum_{m=1}^{k} l_m \leq n\gamma(k+1), \text{ for all } k \leq K-1. \tag{10}$$

Since $\sum_{m=1}^{k} l_m$ has to be integer, the above two bounds lead to an essentially unique distribution of phantoms.

Theorem 2 *Suppose that Assumptions A and B hold, and let $\lceil z \rceil$ denote the largest integer that is below z. The optimal mechanism for n agents is a generalized median scheme with $(n-1)$ phantom voters' peaks distributed according to*

$$l_k^* = \begin{cases} \lceil n\gamma(2) \rceil & \text{if} \quad k = 1 \\ \lceil n\gamma(k+1) \rceil - \lceil n\gamma(k) \rceil & \text{if} \quad 1 < k < K \\ n - 1 - \sum_{m=1}^{K-1} l_m^* & \text{if} \quad k = K \end{cases}.$$

The above theorem reveals that adding or eliminating an alternative has only a local effect. That is, if we add an alternative k_1 such that an interval $\left[x^k, x^{k+1}\right]$ is further divided into $\left[x^k, x^{k_1}\right]$ and $\left[x^{k_1}, x^{k+1}\right]$, the only effect on the optimal phantom allocation is that the original number of phantoms placed on alternatives k and $k+1$ are split between the original alternatives k, $k+1$, and the new alternative k_1. Similarly, if we eliminate alternative k, then the phantoms that were allocated on this alternative are now re-allocated to adjacent alternatives $k-1$ and $k+1$, without any effect on the other alternatives. This "locality-effect" follows from the single-peaked preferences: the social planner does not want to change the chosen alternative if the peak of the median voter does not change as a result of adding/eliminating the available alternatives.

The optimal mechanism introduces biases towards different alternatives, through allocation of phantom voters, that take into account the agents' utilities from the different alternatives and the distribution of types. The above theorem also implies that, with a large number of voters, the optimal mechanism places phantom peaks on every alternative.

[13] This is feasible only if $l_k > 0$. It turns out that the derived bounds (9) and (10) remain valid for alternatives with zero phantom voters. See Lemma 2 in the Appendix.

Remark 4 *What if Assumption B fails? Suppose that* $\Gamma(k+1) < \Gamma(k)$*, and thus that* $\gamma(k) < \gamma(k+1)$*. Then we must have* $l_k^* = 0$*. To see this, suppose by contradiction that* $l_k^* > 0$*. We can apply bounds (9) and (10) to obtain that*

$$l_k^* = \sum_{m=1}^{k} l_m^* - \sum_{m=1}^{k-1} l_m^* \leq n\left[\gamma(k+1) - \gamma(k)\right] + 1 < 1.$$

Since l_k *has to be an integer, we obtain that* $l_k^* = 0$*, a contradiction.*

If there are only two alternatives, Theorem 2 specifies the optimal qualified majority rule. That is, the optimal decision can also be implemented by voting with a properly chosen majority rule. Here are two examples: 1) Zero phantoms on one of the alternatives corresponds to the unanimity rule, and such a rule can be optimal only if the number of the real voters is relatively small; 2) For n odd, $(n-1)/2$ phantoms on each alternative corresponds to the simple majority rule, and such a rule is optimal in symmetric situations. More generally, each optimal rule is a qualified majority rule, where the bias in favor of one alternative depends on the ratio of expected relative losses suffered in each situation by those whose preferred alternative was not chosen. The following corollary characterizes the optimal voting rule in the case of two alternatives. Note that Assumption B is not needed for this result.

Corollary 1 *Suppose there are* n *agents and only two alternatives,* $K = 2$*. Under Assumption A, the optimal rule is implemented through a voting game in which alternative 1 is chosen if and only if at least* $n - \lceil n\gamma(2) \rceil$ *voters voted in its favour.*

Our next result shows that, when the number of voters is large, the optimal (second-best) mechanism approximates the welfare maximizing mechanism (first-best) which, as illustrated in Example 1, is not implementable in our setting. In other words, our optimal mechanism corrects for the difference in the alternatives preferred by the (real) median voter and the one yielding the highest average welfare.

While this result is relatively intuitive (since the aggregate uncertainty vanishes in the limit), the proof is not trivial. It uses the single-crossing and regularity assumption made above. A result in the same spirit for settings with only two alternatives has been obtained by Ledyard and Palfrey [2002].

Theorem 3 *Suppose that Assumptions A and B hold. Let the number of agents* n *go to infinity. Then the optimal DIC mechanism (second-best) converges to the optimal mechanism under complete information (first-best). In other words, in the limit, the optimal DIC mechanism yields the social alternative with the highest expected welfare.*

5.2 The Linear Case

We illustrate our characterization of optimal mechanisms in the linear environment set out in Section 3. For this environment, we introduce a simpler assumption to replace Assumption

B. Let X be the random variable representing the agents' type. We first define two functions, $C(x)$ and $c(x)$, as follows:

$$C(x) = E[X|X > x] \text{ and } c(x) = E[X|X \leq x].$$

Assumption B' *The functions $x - C(x)$ and $x - c(x)$ are increasing.*

In Appendix B we offer sufficient conditions on the distribution of types for Assumption B' to hold. These are related to ubiquitous conditions on hazard rates, well-known from the theory of optimal mechanism design with quasi-linear utility and monetary transfers. In the linear environment, the function $\gamma(k)$ becomes

$$\gamma(k) = \frac{x^k - c\left(x^k\right)}{C\left(x^k\right) - c\left(x^k\right)}.$$

Under Assumption B', the function

$$\frac{C\left(x^k\right) - c\left(x^k\right)}{x^k - c\left(x^k\right)} = 1 - \frac{x^k - C\left(x^k\right)}{x^k - c\left(x^k\right)}$$

is decreasing in x^k. It follows that $\gamma(k)$ is increasing since

$$\gamma(k+1) = \frac{x^{k+1} - c\left(x^{k+1}\right)}{C\left(x^{k+1}\right) - c\left(x^{k+1}\right)} \geq \frac{x^k - c\left(x^k\right)}{C\left(x^k\right) - c\left(x^k\right)} = \gamma(k).$$

Therefore, we obtain the following corollary to Theorem 2.

Corollary 2 *Suppose that Assumptions A and B' hold, and let $\lceil z \rceil$ denote the largest integer that is below z. The optimal mechanism for n agents is a generalized median scheme with $(n-1)$ phantom voters' peaks distributed according to*

$$l_k^* = \begin{cases} \left\lceil n \frac{x^2 - c(x^2)}{C(x^2) - c(x^2)} \right\rceil & \text{if} \quad k = 1 \\ \left\lceil n \frac{x^{k+1} - c(x^{k+1})}{C(x^{k+1}) - c(x^{k+1})} \right\rceil - \left\lceil n \frac{x^k - c(x^k)}{C(x^k) - c(x^k)} \right\rceil & \text{if} \quad 1 < k < K \\ n - 1 - \sum_{m=1}^{K-1} l_m^* & \text{if} \quad k = K \end{cases}.$$

This corollary yields immediate and intuitive comparative statics with respect to parameters of the linear utility function $\{a_k, b_k\}_{k=1}^K$. By the definition of the cutoffs x^k, increases in either a_k or b_k decrease x^k and increase x^{k+1}, which in turn increase l_k. That is, if the attractiveness of any alternative increases, the optimal number of the phantom voters with peaks on this alternative increases as well.

Example 4 *Suppose that the distribution of signals F is uniform on $[0,1]$. Then $C(x) = E[X|X > x] = (1+x)/2$ and $c(x) = E[X|X \leq x] = x/2$. Therefore, the optimal distribution of phantom voters' peaks is given by: $l_k^* = \lceil nx^{k+1} \rceil - \lceil nx^k \rceil$. Intuitively, here the number of phantom voters' peaks is proportional to the share of real types whose top alternative is k.*

We can also further illustrate the above corollary by describing in more detail the optimal voting rules when there are two agents. If there are two agents (i and $-i$) and K alternatives, the set of peaks-only, onto, DIC and anonymous mechanisms contains exactly K generalized median voter schemes with one phantom voter (see Theorem 1). Therefore, we only need to find the optimal position for the one additional phantom voter.

Corollary 3 *Suppose there are only two agents. Under Assumptions A and B', the optimal mechanism is a generalized median voter scheme with one phantom voter whose peak is placed on*

$$k^* \equiv \min \left\{ k \in \mathcal{K} : x^{k+1} \geq \frac{1}{2} \left[C\left(x^{k+1}\right) + c\left(x^{k+1}\right) \right] \right\}.$$

Note that the condition for determining the optimal phantom voter peak can be rewritten as $k^* = k$ if $x^* \in \left[x^k, x^{k+1}\right]$, where $x^* = \frac{1}{2}\left[C\left(x^*\right) + c\left(x^*\right)\right]$. The critical value x^* has the same distance to the upper conditional mean $C\left(x^*\right)$ as to the lower conditional mean $c\left(x^*\right)$. In particular, if the distribution is symmetric, then x^* coincides with the mean of the distribution.

5.3 Other Objective Functions

Other, non-utilitarian, objective functions can be considered as well. For example, if the designer's preferences are *maximin*, then the desired allocation is

$$g^{\min}\left(x_1, ..., x_n\right) = k^m$$

where k^m satisfies $x^m \in (x^{k^m}, x^{k^m+1}]$ with $x^m = \min\{x_1, ..., x_n\}$. That is, k^m is the most preferred alternative of the agent with the lowest signal. This rule is implementable through a peaks-only mechanism

$$\pi^{\min}\left(k_1, ..., k_n\right) = \min\{k_1, ..., k_n\}.$$

In such a case, all phantoms should be allocated on alternative 1.

Similarly, if the designer's preferences are *maximax*, then the designer would like to implement allocation

$$g^{\max}\left(x_1, ..., x_n\right) = k^M$$

where k^M satisfies $x^M \in (x^{k^M}, x^{k^M+1}]$ with $x^M = \max\{x_1, ..., x_n\}$. That is, k^M is the most preferred alternative of the agent with the highest signal, and this rule is also implementable through a peaks-only mechanism

$$\pi^{\max}\left(k_1, ..., k_n\right) = \max\{k_1, ..., k_n\}.$$

In such a case, all phantoms should be allocated on alternative K.

6 Concluding Remarks

We have characterized constrained efficient (i.e., second-best) dominant strategy incentive compatible and deterministic mechanisms in a setting where privately informed agents have single-crossing utility functions, but where monetary transfers are not feasible. Our approach allows a systematic choice among Pareto-efficient mechanisms based on the ex-ante utility they generate.

In the standard setting with independent types, linear utility and monetary transfers, an equivalence result between dominant strategy incentive compatible and Bayes-Nash incentive compatible mechanisms has been established by Gershkov et al. [2013]. Dominant strategy mechanisms are robust to variations in beliefs. It is an open question whether using the more permissible Bayesian incentive compatibility concept can improve the performance of constrained efficient mechanisms also in settings without monetary transfers.

7 Appendix A: Proofs

In the proof of Lemma 1, we will use the following definition and a proposition due to Saporiti [2009].

Definition 2 *A mechanism π is tops-only if it has the form $\pi : \mathcal{K}^n \to \mathcal{K}$. We say that a DIC mechanism g is equivalent to a tops-only mechanism π if*

$$g(x_1, ..., x_n) = \pi(k_1, ..., k_n),$$

for any type profile $(x_1, ..., x_n)$ and for any alternative profile $(k_1, ..., k_n)$ such that $x_i \in (x^{k_i}, x^{k_i+1}]$ for all i

Proposition 1 (Saporiti 2009) *A mechanism g is DIC only if it is equivalent to a tops-only mechanism.*

Proof of Lemma 1. We first show that every onto and DIC mechanism satisfies unanimity. We prove it by contradiction. Suppose there exist an alternative k and a report profile $(\widehat{x}_1, ..., \widehat{x}_n)$ such that $\widehat{x}_i \in \left(x^k, x^{k+1}\right)$ for all i but $g(\widehat{x}_1, ..., \widehat{x}_n) = l$ with $l \neq k$. Since the mechanism is onto, there exists some type profile $(x_1^*, ..., x_n^*)$ such that $g(x_1^*, ..., x_n^*) = k$. First suppose $x_i^* \in \left(x^k, x^{k+1}\right)$ for all i. Consider agent 1 and fix the other agents' reports at $(x_2^*, ..., x_n^*)$. DIC implies that $g(\widehat{x}_1, x_2^*, ..., x_n^*) = k$, otherwise agent 1 could manipulate at $(\widehat{x}_1, x_2^* ..., x_n^*)$ via x_1^* to achieve his best alternative k. Next consider agent 2, and fix the other agents' reports at $(\widehat{x}_1, x_3^*, ..., x_n^*)$. Then, again we must have $g(\widehat{x}_1, \widehat{x}_2, x_3^*, ..., x_n^*) = k$, otherwise agent 2 could manipulate at $(\widehat{x}_1, \widehat{x}_2, x_3^*, ..., x_n^*)$ via x_2^*. Applying the same argument to the remaining agents, $3, ..., n$, we obtain that $g(\widehat{x}_1, ..., \widehat{x}_n) = k$, which is a contradiction. Therefore, there must exist at least one agent i such that $x_i^* \notin \left(x^k, x^{k+1}\right)$ and $g(x_1^*, ..., x_i^{**}, ..., x_n^*) = m$ with $m \neq k$ and $x_i^{**} \in \left(x^k, x^{k+1}\right)$. Fix the reports of all agents but

i to $\left(x_1^*, ..., x_{i-1}^*, x_{i+1}^*, ..., x_n^*\right)$. This mechanism is not incentive compatible, because agent i with type x_i^{**} could manipulate at $\left(x_1^*, ..., x_{i-1}^*, x_i^{**}, x_{i+1}^*, ..., x_n^*\right)$ via x_i^* and achieve his best alternative k.

We now show that every onto and DIC mechanism is Pareto efficient. Due to the last Proposition, it is sufficient to prove the statement for tops-only mechanisms. Consider any deterministic DIC and onto mechanism $\pi(k_1, ..., k_n)$ where $k_1, ..., k_n$ are the reported peaks. The Pareto set given peaks $(k_1, ..., k_n)$ is

$$\{k \in \mathcal{K} : \min(k_1, ..., k_n) \leq k \leq \max(k_1, ..., k_n)\}.$$

Consider any profile of peaks $\left(\widehat{k}_1, ..., \widehat{k}_n\right)$. In order to show that π is Pareto efficient, it is sufficient to show that,

$$\min\left(\widehat{k}_1, ..., \widehat{k}_n\right) \leq \pi\left(\widehat{k}_1, ..., \widehat{k}_n\right) \leq \max\left(\widehat{k}_1, ..., \widehat{k}_n\right).$$

We prove the above by contradiction. First assume that $\pi\left(\widehat{k}_1, ..., \widehat{k}_n\right) > k \equiv \max\left(\widehat{k}_1, ..., \widehat{k}_n\right)$. Since the mechanism is onto, there exists a profile $(k_1^*, ..., k_n^*)$ such that

$$\pi(k_1^*, ..., k_n^*) = k.$$

Consider agent 1, and fix the types of other agents at $\widehat{k}_{-1} = \left(\widehat{k}_2, ..., \widehat{k}_n\right)$. Then DIC for agent 1 with type $\widehat{k}_1$ implies that, for all k_1, $\pi\left(k_1, \widehat{k}_2, ..., \widehat{k}_n\right) > k$. Now fix agent 1's type at k_1^*, and consider agent 2. Since $\pi\left(k_1^*, \widehat{k}_2, \widehat{k}_3, ..., \widehat{k}_n\right) > k$, DIC for agent 2 with type $\widehat{k}_2$ implies that, for all k_2, $\pi\left(k_1^*, k_2, \widehat{k}_3, .., \widehat{k}_n\right) > k$. Now we fix agent 1 and 2's types at k_1^*, k_2^*, respectively. We can proceed as before and consider agent 3. We can argue that for all k_3, we have $\pi\left(k_1^*, k_2^*, k_3, \widehat{k}_4, .., \widehat{k}_n\right) > k$. Therefore for all k_n, we have $\pi\left(k_1^*, .., k_{n-1}^*, k_n\right) > k$. But this contradicts the fact that $\pi\left(k_i^*, k_{-i}^*\right) = k$.

The proof of $\pi\left(\widehat{k}_1, ..., \widehat{k}_n\right) \geq \min\left(\widehat{k}_1, ..., \widehat{k}_n\right)$ is similar. Therefore, any deterministic DIC and onto mechanism must be Pareto efficient. ■

Proof of Theorem 2. Let $k \geq 2$, and suppose that $l_k > 0$ is part of the optimal allocation of $(n-1)$ phantoms. By optimality, the social planner must prefer this allocation of phantoms over allocating $l_k - 1$ phantoms on alternative k and $l_{k-1} + 1$ phantoms on alternative $k-1$. This change matters only if it affects the median among $n-1$ phantom and n real voters. For this to happen, it must be that the total number of voters ("real" and "phantom") with values below x^k is $(n-1)$: there are exactly $\left(n - 1 - \sum_{m=1}^{k-1} l_m\right)$ "real" voters with values below x^k and $\left(\sum_{m=1}^{k-1} l_m + 1\right)$ "real" voters with values above x^k. In this case, by moving a phantom from alternative k to alternative $k-1$, the planner changes the median from k to $k-1$. In this case, the total expected utility from alternative k is given by

$$\left(n - 1 - \sum_{m=1}^{k-1} l_m\right) u_{x<x^k}^k + \left(\sum_{m=1}^{k-1} l_m + 1\right) u_{x>x^k}^k.$$

The total expected utility from alternative $k-1$ is given by

$$\left(n-1-\sum_{m=1}^{k-1} l_m\right) u_{x<x^k}^{k-1} + \left(\sum_{m=1}^{k-1} l_m + 1\right) u_{x>x^k}^{k-1}.$$

Since the planner (weakly) prefers k to $k-1$, the total expected utility from alternative k must be higher than the total expected utility from alternative $k-1$. This gives us the following "first-order condition" for all $k \geq 2$ with $l_k > 0$:

$$\left(n-1-\sum_{m=1}^{k-1} l_m\right)\left(u_{x<x^k}^{k} - u_{x<x^k}^{k-1}\right) + \left(\sum_{m=1}^{k-1} l_m + 1\right)\left(u_{x>x^k}^{k} - u_{x>x^k}^{k-1}\right) \geq 0 \tag{11}$$

Similarly, if $l_k > 0$ with $k \leq K-1$ is part of the optimal allocation of $(n-1)$ phantoms, then the social planner must prefer this allocation of phantoms to allocating $l_k - 1$ phantoms on alternative k and $l_{k+1}+1$ phantoms on alternative $k+1$. This change matters only if it affects the median among $n-1$ phantom and n real voters. For this to happen, it must be that the total number of voters ("real" and "phantom") with values below x^{k+1} is n. In other words, there are exactly $\left(n-\sum_{m=1}^{k} l_m\right)$ "real" voters with values below x^{k+1} and $\sum_{m=1}^{k-1} l_m$ "real" voters with values above x^{k+1}. In this case, the total expected utility from alternative k is given by

$$\left(n-\sum_{m=1}^{k} l_m\right) u_{x<x^{k+1}}^{k} + \left(\sum_{m=1}^{k} l_m\right) u_{x>x^{k+1}}^{k}.$$

The total expected utility from alternative $k-1$ is given by

$$\left(n-\sum_{m=1}^{k} l_m\right) u_{x<x^{k+1}}^{k+1} + \left(\sum_{m=1}^{k} l_m\right) u_{x>x^{k+1}}^{k+1}.$$

This yields another "first-order condition" for all $k \leq K-1$ with $l_k > 0$:

$$\left(n-\sum_{m=1}^{k} l_m\right)\left(u_{x<x^{k+1}}^{k} - u_{x<x^{k+1}}^{k+1}\right) + \left(\sum_{m=1}^{k} l_m\right)\left(u_{x>x^{k+1}}^{k} - u_{x>x^{k+1}}^{k+1}\right) \geq 0. \tag{12}$$

These two first-order conditions can be rewritten as bounds on phantom distributions for alternatives k with $l_k > 0$. These were inequalities (9) and (10) from the main text, which we reproduce below:

$$\begin{aligned}
\sum_{m=1}^{k-1} l_m &\geq \frac{n\left(u_{x<x^k}^{k-1} - u_{x<x^k}^{k}\right)}{\left(u_{x<x^k}^{k-1} - u_{x<x^k}^{k}\right) + \left(u_{x>x^k}^{k} - u_{x>x^k}^{k-1}\right)} - 1 = \gamma(k) - 1, \text{ for all } k \geq 2, \\
\sum_{m=1}^{k} l_m &\leq \frac{n\left(u_{x<x^{k+1}}^{k} - u_{x<x^{k+1}}^{k+1}\right)}{\left(u_{x<x^{k+1}}^{k} - u_{x<x^{k+1}}^{k+1}\right) + \left(u_{x>x^{k+1}}^{k+1} - u_{x>x^{k+1}}^{k}\right)} = \gamma(k+1), \text{ for all } k \leq K-1.
\end{aligned}$$

Lemma 2 below shows that the above two conditions hold for alternative k with $l_k = 0$.

Therefore, we can construct the (unique) candidate distribution of phantom voters' peaks as follows. We first derive bounds for l_1^* by taking $k = 2$ in (9) and $k = 1$ in (10):

$$n\gamma(2) - 1 \leq l_1^* \leq n\gamma(2).$$

Since the two bounds differ by 1 and l_1^* must be an integer, l_1^* is generically unique and must be equal to $\lceil n\gamma(2)\rceil$,where $\lceil z\rceil$ denotes the largest integer that is below z.

Next note that, for all $2 \leq k \leq K-1$, conditions (9) and (10) imply that

$$n\gamma(k+1) - 1 \leq \sum_{m=1}^{k-1} l_m^* \leq n\gamma(k+1).$$

Hence, $\sum_{m=1}^{k} l_m^*$ is also generically unique and must be equal to $\lceil n\gamma(k+1)\rceil$ .As a result, we can deduce l_2^* as

$$l_2^* = \sum_{m=1}^{2} l_m^* - l_1^* = \lceil n\gamma(3)\rceil - \lceil n\gamma(2)\rceil.$$

Similarly, we can obtain recursively for all l_k^* with $2 \leq k \leq K-1$:

$$l_k^* = \lceil n\gamma(k+1)\rceil - \lceil n\gamma(k)\rceil.$$

Since by Assumption B, $\gamma(k)$ is increasing in k. Hence, we obtain that $l_k^* \geq 0$.

Finally, since there are $(n-1)$ phantom voters in total, we have

$$l_K^* = n - 1 - \sum_{m=1}^{K-1} l_m^* = n - 1 - \lceil n\gamma(K)\rceil.$$

It is clear that $\gamma(K) < 1$, so $l_K^* \geq 0$.

To complete the proof, we need to argue that the phantom distribution we constructed above is indeed optimal. Note that we are optimizing a bounded function over a discrete domain, so that the optimal solution always exists. Because the optimal solution has to satisfy the two necessary conditions (9) and (10), and because there is essentially unique distribution that satisfies these two conditions, our candidate distribution $\{l_k^*\}$ must be optimal. ■

Lemma 2 *The bounds (9) and (10) hold for all $k \in \mathcal{K}$ with $l_k = 0$.*

Proof. First let us define κ_1 and κ_2 as follows:

$$\begin{aligned} \kappa_1 &= \max\{m \in \mathcal{K} : l_k = 0 \text{ for all } k \leq m\}, \\ \kappa_2 &= \min\{m \in \mathcal{K} : l_k = 0 \text{ for all } k \geq m\}. \end{aligned}$$

We need to consider several cases.

Case 1: Both κ_1 and κ_2 exist. Then we have $l_1 = ... = l_{\kappa_1} = 0$, and $l_{\kappa_2} = ... = l_K = 0$. An alternative k with $l_k = 0$ could belong to one of the following three possible scenarios:

(i) $k \leq \kappa_1$. Since $l_1 = ... = l_{\kappa_1} = 0$, condition (10) holds trivially and we only need to prove condition (9). By definition of κ_1, $l_{\kappa_1+1} > 0$. Thus, (9) must hold at $\kappa_1 + 1$:

$$\sum_{m=1}^{\kappa_1} l_m \geq n\gamma(\kappa_1 + 1) - 1.$$

Since $l_1 = ... = l_{\kappa_1} = 0$, we have

$$\sum_{m=1}^{k-1} l_m = \sum_{m=1}^{\kappa_1} l_m \geq n\gamma(\kappa_1 + 1) - 1 \geq n\gamma(k) - 1,$$

where the second inequality follows because γ is increasing and $\kappa_1 + 1 > k$.

(ii) $k \geq \kappa_2$. Since $l_{\kappa_2} = ... = l_K = 0$, for all $k \geq \kappa_2$, we have

$$\sum_{m=1}^{k-1} l_m = n - 1 - \sum_{k}^{K} l_m = n - 1.$$

Hence, condition (9) is trivially satisfied, and we only need to prove condition (10). By definition of κ_2, $l_{\kappa_2-1} > 0$. So we have (10) hold at $\kappa_2 - 1$:

$$\sum_{m=1}^{\kappa_2-1} l_m \leq n\gamma(\kappa_2).$$

Therefore,

$$\sum_{m=1}^{k} l_m = n - 1 = \sum_{m=1}^{\kappa_2-1} l_m \leq n\gamma(\kappa_2) \leq n\gamma(k+1).$$

Again the last inequality follows from the monotonicity of $\gamma(\cdot)$ and the fact that $\kappa_2 < k + 1$.

(iii) $k \in (\kappa_1, \kappa_2)$. Define k_1 and k_2 as follows:

$$\begin{aligned} k_1 &= \max\{m \in \mathcal{K} : m < k \text{ and } l_m > 0\}, \\ k_2 &= \min\{m \in \mathcal{K} : m > k \text{ and } l_m > 0\}. \end{aligned}$$

Both k_1 and k_2 are well defined for all $k \in (\kappa_1, \kappa_2)$. By definition of k_1 and k_2, we have

$$\sum_{m=1}^{k} l_m = \sum_{m=1}^{k_1} l_m \text{ and } \sum_{m=1}^{k-1} l_m = \sum_{m=1}^{k_2-1} l_m,$$

and condition (9) holds at $k_2 - 1$ and (10) holds at k_1 :

$$\sum_{m=1}^{k_2-1} l_m \geq n\gamma(k_2) - 1, \text{ and } \sum_{m=1}^{k_1} l_m \leq n\gamma(k_1 + 1).$$

Since $\gamma(\cdot)$ is increasing and $k_1 < k < k_2$, we have

$$\sum_{m=1}^{k-1} l_m \geq n\gamma(k) - 1, \text{ and } \sum_{m=1}^{k} l_m \leq n\gamma(k+1).$$

Case 2: Neither κ_1 nor κ_2 exists. Then the argument of Case 1(iii) applies for all k with $l_k = 0$.

Case 3: κ_1 exists but κ_2 does not. Consider alternative k with $l_k = 0$. If $k \leq \kappa_1$, the argument of Case 1(i) applies. If $k > \kappa_1$, the argument of Case 1 (iii) applies.

Case 4: κ_2 exists but κ_1 does not. Consider alternative k with $l_k = 0$. If $k \geq \kappa_2$, the argument of Case 1(ii) applies. If $k < \kappa_2$, the argument of Case 1(iii) applies. ■

Proof of Theorem 3. In the first-best rule the designer chooses alternative k if and only if it maximizes the sum of agents' expected utilities, that is, for any $l \in \mathcal{K} = \{1, ..., K\}$

$$\sum_{i=1}^{n} u_k(x_i) \geq \sum_{i=1}^{n} u_l(x_i) \Longleftrightarrow \sum_{i=1}^{n} \frac{1}{n} u_k(x_i) \geq \sum_{i=1}^{n} \frac{1}{n} u_l(x_i).$$

Let now n go to infinity. Using the law of large numbers, the last inequality becomes

$$\forall l, \int_0^1 u_k(x) f(x) dx \geq \int_0^1 u_l(x) f(x) dx \Longleftrightarrow \int_0^1 (u_k(x) - u_l(x)) f(x) dx \geq 0.$$

Claim: Under Assumptions A and B, an alternative k maximizes the expected social welfare, $k \in \max_{l \in \mathcal{K}} \int_0^1 u_l(x) f(x) dx$, if and only if the following two "local" first-order conditions holds:

$$\int_0^1 (u_k(x) - u_{k-1}(x)) f(x) dx \geq 0 \tag{13}$$

$$\int_0^1 (u_k(x) - u_{k+1}(x)) f(x) dx \geq 0 \tag{14}$$

The necessity is obvious. To show sufficiency, we first prove that if condition (14) holds, then any upward deviation is not profitable: for all $l > k$,

$$\int_0^1 (u_k(x) - u_l(x)) f(x) dx \geq 0.$$

We first notice that

$$\begin{aligned} & \int_0^1 (u_k(x) - u_{k+1}(x)) f(x) dx \\ = & \int_0^{x^{k+1}} (u_k(x) - u_{k+1}(x)) f(x) dx + \int_{x^{k+1}}^1 (u_k(x) - u_{k+1}(x)) f(x) dx \\ = & F\left(x^{k+1}\right) \left(u^k_{x<x^{k+1}} - u^{k+1}_{x<x^{k+1}}\right) + \left(1 - F\left(x^{k+1}\right)\right) \left(u^k_{x>x^{k+1}} - u^{k+1}_{x>x^{k+1}}\right). \end{aligned}$$

Therefore, condition (14) is equivalent to

$$\Gamma(k+1) = \frac{u^k_{x<x^{k+1}} - u^{k+1}_{x<x^{k+1}}}{u^{k+1}_{x>x^{k+1}} - u^k_{x>x^{k+1}}} \geq \frac{1 - F\left(x^{k+1}\right)}{F\left(x^{k+1}\right)}.$$

By definition of cutoffs and Assumption B, $x^{k+2} > x^{k+1}$ and $\Gamma(k+1) \leq \Gamma(k+2)$. Therefore, condition (14) implies

$$\frac{u^{k+1}_{x<x^{k+2}} - u^{k+2}_{x<x^{k+2}}}{u^{k+2}_{x>x^{k+2}} - u^{k+1}_{x>x^{k+2}}} \geq \frac{u^{k}_{x<x^{k+1}} - u^{k+1}_{x<x^{k+1}}}{u^{k+1}_{x>x^{k+1}} - u^{k}_{x>x^{k+1}}} \geq \frac{1 - F\left(x^{k+1}\right)}{F\left(x^{k+1}\right)} \geq \frac{1 - F\left(x^{k+2}\right)}{F\left(x^{k+2}\right)}$$

which in turn implies

$$F\left(x^{k+2}\right)\left(u^{k+1}_{x<x^{k+2}} - u^{k+2}_{x<x^{k+2}}\right) + \left(1 - F\left(x^{k+2}\right)\right)\left(u^{k+1}_{x>x^{k+2}} - u^{k+2}_{x>x^{k+2}}\right) \geq 0.$$

The last inequality can be rewritten as

$$\int_0^1 \left(u_{k+1}(x) - u_{k+2}(x)\right) f(x)dx \geq 0.$$

By applying the above logic recursively, we have proved that (14) implies, for all $k < l \leq K$,

$$\int_0^1 \left(u_l(x) - u_{l+1}(x)\right) f(x)dx \geq 0. \tag{15}$$

Next we note that, if condition (14) holds, then for all $k < l \leq K$,

$$\begin{aligned}
& \int_0^1 \left(u_k(x) - u_l(x)\right) f(x)dx \\
= & \int_0^1 \left(u_k(x) - u_{k+1}(x) + u_{k+1}(x) - \ldots - \ u_{l-1}(x) + u_{l-1}(x) - u_l(x)\right) f(x)dx \\
= & \int_0^1 \left(u_k(x) - u_{k+1}(x)\right) f(x)dx + \int_0^1 \left(u_{k+1}(x) - u_{k+2}(x)\right) f(x)dx \\
& + \ldots + \int_0^1 \left(u_{l-1}(x) - u_l(x)\right) f(x)dx \\
\geq & \ 0
\end{aligned}$$

The last inequality follows because each integral is positive by using (15) repeatedly.

We can follow an analogous procedure to show that if condition (13) holds, then any downward deviation is not profitable: for all $1 \leq l < k$,

$$\int_0^1 \left(u_k(x) - u_l(x)\right) f(x)dx \geq 0.$$

This concludes the proof of the above Claim.

It follows from the above Claim that the first-best, welfare maximizing alternative is the lowest alternative k for which

$$\int_0^1 \left(u_k(x) - u_{k+1}(x)\right) f(x)dx \geq 0 \Longleftrightarrow \frac{u^{k}_{x<x^{k+1}} - u^{k+1}_{x<x^{k+1}}}{u^{k+1}_{x>x^{k+1}} - u^{k}_{x>x^{k+1}}} \geq \frac{1 - F\left(x^{k+1}\right)}{F\left(x^{k+1}\right)}$$

In the optimal DIC mechanism (second-best) alternative k is chosen if it is the lowest alternative such that the number of voters (both real and phantom) with peaks on alternatives below k is at least n (recall that the total number of voters, real and phantom, is $2n-1$). In the limit where n goes to infinity alternative k is chosen if it is the lowest alternative such that

$$\frac{u^k_{x<x^{k+1}} - u^{k+1}_{x<x^{k+1}}}{u^{k+1}_{x>x^{k+1}} - u^k_{x>x^{k+1}} + u^k_{x<x^{k+1}} - u^{k+1}_{x<x^{k+1}}} + F\left(x^{k+1}\right) \geq 1 \Longleftrightarrow \frac{u^k_{x<x^{k+1}} - u^{k+1}_{x<x^{k+1}}}{u^{k+1}_{x>x^{k+1}} - u^k_{x>x^{k+1}}} \geq \frac{1-F\left(x^{k+1}\right)}{F\left(x^{k+1}\right)}$$

Thus, we obtain the same rule as in the first-best mechanism, as desired. ■

Proof of Corollary 3. Recall that the candidate position k^* is defined as

$$k^* \equiv \min\left\{k \in \mathcal{K} : x^{k+1} \geq (C(x^{k+1}) + c(x^{k+1}))/2\right\}.$$

By definition of k^*

$$x^{k+1} \geq (c\left(x^{k+1}\right) + C\left(x^{k+1}\right))/2 \text{ for all } k \geq k^*,$$

and

$$x^{k+1} < (c\left(x^{k+1}\right) + C\left(x^{k+1}\right))/2 \text{ for all } k < k^*.$$

This implies that

$$\frac{x^{k+1} - c(x^{k+1})}{C(x^{k+1}) - c(x^{k+1})} \geq 1/2, \text{ for all } k \geq k^*, \tag{16}$$

and

$$\frac{x^{k+1} - c(x^{k+1})}{C(x^{k+1}) - c(x^{k+1})} < 1/2 \text{ for all } k < k^*. \tag{17}$$

Moreover we note that for all k

$$\frac{x^{k+1} - c(x^{k+1})}{C(x^{k+1}) - c(x^{k+1})} < 1.$$

Therefore, in order to satisfy both (16) and (17), we must have, in the optimal phantom distribution, $l^*_{k^*} = 1$, and $l^*_k = 0$ for all $k \neq k^*$. ■

8 Appendix B: Sufficient Conditions for Assumption B and Assumption B'

To derive sufficient conditions for Assumption B, we first introduce several well known definitions of stochastic orders (see Shaked and Shanthikumar [2007]).

Definition 3 *(i) A random variable Y dominates a random variable X in the usual stochastic order (or first-order stochastic dominance, denoted by $X \leq_{st} Y$) if* $\Pr\{X > x\} \leq \Pr\{Y > x\}$ *for all x.*

(ii) A random variable Y dominates a random variable X in the hazard rate order (denoted as $X \leq_{hr} Y$) if $[X|X > x] \leq_{st} [Y|Y > x]$ for all x.

(iii) A random variable Y dominates a random variable X in the reverse hazard rate order (denoted as $X \leq_{rh} Y$) if $[X|X < x] \leq_{st} [Y|Y < x]$ for all x.

(iv) A random variable Y dominates a random variable X in the likelihood ratio order (denoted as $X \leq_{lr} Y$) if $[X|a \leq X \leq b] \leq_{st} [Y|a \leq Y \leq b]$ for all $a < b$.

It is clear from the above definitions that $X \leq_{lr} Y$ implies both $X \leq_{hr} Y$ and $X \leq_{rh} Y$. We now let $h_k(x)$ denote the utility difference for a type-x agent from two adjacent alternatives k and $k-1$:

$$h_k(x) = u^{k-1}(x) - u^k(x).$$

We claim that if the random variables $\{h_k(x)\}_{k \in \mathcal{K}}$ are ordered in terms of both hazard rate order and reverse hazard rate order, that is, $h_k \leq_{hr} h_{k+1}$ and $h_k \leq_{rh} h_{k+1}$, then Assumption B holds.[14] To see this, note that we can write

$$u^{k-1}_{x<x^k} - u^k_{x<x^k} = E[h_k(x) \mid x < x^k] = E[h_k(x) \mid h_k(x) > 0]$$

where the second equality follows from the definition of cutoff x^k and the single-crossing property. By rewriting $u^k_{x>x^k} - u^{k-1}_{x>x^k}$ analogously, we obtain

$$\Gamma(k) = -\frac{E[h_k(x) \mid h_k(x) > 0]}{E[h_k(x) \mid h_k(x) < 0]}.$$

Note that $h_k \leq_{hr} h_{k+1}$ implies that $E[h_k(x) \mid h_k(x) > 0]$ is increasing in k, and $h_k \leq_{rh} h_{k+1}$ implies that $E[h_k(x) \mid h_k(x) < 0]$ is increasing in k. Therefore, $\Gamma(k)$ is increasing in k.

To derive sufficient conditions for Assumption B' to hold in the linear case, let us first recall a well-known concept used in the theory of reliability.

Definition 4 *(i) The mean residual life (MRL) of a random variable $X \in [0, \overline{\theta}]$ is defined as*

$$MRL(x) = \begin{cases} E[X - x | X \geq x] & \text{if} \quad x < \overline{\theta} \\ 0 & \text{if} \quad x = \overline{\theta} \end{cases}$$

(ii) A random variable X satisfies the decreasing mean residual life (DMRL) property if the function $MRL(x)$ is decreasing in x.

If we let X denote the life-time of a component, then $MRL(x)$ measures the expected remaining life of a component that has survived until time x.[15] Assuming that $x - C(x)$ is

[14] Note that conditions $h_k \leq_{hr} h_{k+1}$ and $h_k \leq_{rh} h_{k+1}$ impose restrictions on the shapes of both the distribution F and the utility function u. Alternatively, if we assume F is uniform, we could explicitly derive the required conditions for Assumption B only on function u. On the other hand, if we assume that the utility function u is linear as in our linear case that will be investigated below, we can derive the required conditions for Assumption B only on the distribution F.

[15] The MRL function is related to the hazard rate (or failure rate) $\lambda(x) = f(x) / [1 - F(x)]$. The "increasing failure rate" (IFR) assumption is commonly made in the economics literature. DMRL is a weaker property, and it is implied by IFR.

increasing is equivalent to assuming a decreasing mean residual life (DMRL). Assuming that $x - c(x)$ is increasing is equivalent to assuming log-concavity of $\int_0^x F(s)\,ds$, because

$$x - c(x) = \frac{\int_0^x F(s)\,ds}{F(x)} \quad \text{and} \quad \frac{F(x)}{\int_0^x F(s)\,ds} = \frac{d}{dx}\log\left[\int_0^x F(s)\,ds\right]$$

A sufficient condition for $\int_0^x F(s)\,ds$ to be log-concave is that $F(x)$ is log-concave. A sufficient condition for both log-concavity of F and DMRL of F is that the density f is log-concave.[16]

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[16] The log-concavity of density is stronger than (and implies) increasing failure rate (IFR) which is equivalent to log-concavity of the reliability function $(1-F)$. The family of log-concave densities is large and includes many commonly used distributions such as uniform, normal, exponential, logistic, extreme value etc. The power function distribution $\left(F(x) = x^k\right)$ has log-concave density if $k \geq 1$, but it does not if $k < 1$. However, one can easily verify the above two conditions hold for $F(x) = x^k$ even with $k < 1$. Therefore, log-concave density is not necessary. See Bagnoli and Bergstrom [2005] for an excellent discussion of log-concave distributions.

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