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Market Share Dynamics in a Model with Search and Word-of-Mouth Communication

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Abstract

This paper analyzes price competition in an infinitely repeated duopoly game. In each period, consumers remember the existence and location of their previous supplier. New information is gathered via search or word-of-mouth communication. Market outcomes are history-dependent, and the Markov perfection refinement is used to narrow the set of equilibria. Firms are shown to use mixed pricing strategies in equilibrium. The resulting price dispersion generates non-trivial market share dynamics. The goal of the paper is to characterize these dynamics, and to reveal the driving forces behind them.

Keywords: repeat purchasing; search; customer loyalty; lock-in; mixed pricing

JEL classification: D43, D83, L11

1. Introduction

In markets to which consumers return repeatedly over time ("repeat purchasing"), information gathered in previous periods can affect consumers' behavior also in future periods. E.g., a consumer may remember the existence or location of a supplier visited in the past, while an alternative supplier can be discovered via search, or by communicating with other consumers ("word-of-mouth communication"). Unless all consumers search, some may remain locked-in at their previous supplier.[1] Hence, a firm with a larger customer base (with a high sales volume in the past) is in an advantageous position, as it can exercise monopoly power over those consumers in its customer base who do not discover an alternative supplier.

This paper analyzes dynamic price competition in a homogeneous goods duopoly that matches this description. Each period, a generalized Bertrand game is played where demand depends not only on current prices but also on the previous market share of a firm. Hence, market outcomes are history-dependent. The difference to a static environment is, that firms have an incentive to *invest* in the size of their customer base, as it affects their future profitability.[2] A typical feature of (static) search models with homogeneous goods is the non-existence of pure strategy equilibria. Mixed pricing strategies, thus, lead to price dispersion.[3] The model introduced in this paper also has this property. However, in contrast to most of the existing models, the history-dependence of market outcomes creates inertia in the distribution of market shares. In conjunction with the endogenous[4] price dispersion, this leads to rather non-trivial market share dynamics. The goal of this paper is to characterize these dynamics, and to reveal the driving forces behind them.

The idea that mixed pricing strategies can generate market share dynamics is not new. Chen and Rosenthal (1996), e.g., introduce an infinitely repeated pricing game where a firm that currently offers the higher price loses market shares compared to the previous period, but its market share does not drop to zero (unless it was zero or just above zero before). However, these authors assume that a firm that looses (or gains) market shares always looses the *same* number of customers, irrespective of its previous market share. Hence, there is a uniform 'step size' in the market share space, an assumption that seems poorly justified.[5]

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[1] An alternative explanation are switching costs. For an overview over this literature, see Klemperer (1995).

[2] Authors from the search literature usually adopt a static modeling framework. See Janssen and Moraga-Gonzalez (2004), and the references cited therein. Since consumers (by assumption) do not possess any prior information, authors often assume that consumers randomize over the first supplier they visit. Hence, they focus on *ex-ante* symmetric splits of a market, while the *ex-post* split is usually asymmetric.

[3] The prevalence and persistence of price dispersion is demonstrated empirically by Lach (2002).

[4] There are no external sources of uncertainty or noise in the model.

[5] In his seminal paper, Selten (1965) introduced the concept of 'demand inertia', according to which a firm's current demand depends on its current price and the average price of its competitors, as well as on its last-period

The present paper offers a microeconomic foundation for market share dynamics generated by mixed pricing equilibria. The details of the model are as follows. Each consumer either purchases zero or one unit of the homogeneous good in each period. A consumer who returns to the market to make a purchase, remembers the existence / the location of the previous supplier.[6] Given this information, a current price quote can be obtained for free. An alternative supplier can be discovered via search or word-of-mouth communication. Consumers who discover both suppliers purchase from the one that currently offers the lower price. There are three types of consumers with different search costs. Consumers of the first type (the "searchers") have zero search costs and can, thus, compare prices at no cost.[7] Consumers of the second type ("word-of-mouth consumers") face high search costs, but can communicate with one other consumer for free.[8,9] If the consumer visited the other supplier in the last period, the word-of-mouth consumer discovers this supplier.[10] Otherwise, he/she remains locked-in at her previous supplier. Consumers of the third type ("ignorant type") always remain locked-in at their previous supplier.[11]

There are several aspects that affect the dynamics in this game. First of all, there are the properties that are inherent to the process of information transmission. E.g., if most consumers are of the 'ignorant type', a firm that chooses the lower price in the current period can only attract a small number of new customers. Hence, market share dynamics are characterized by a high degree of inertia. Conversely, if most consumers are searchers, the low-price firm serves almost the entire market, irrespective of the size of its customer base. Hence, market shares are volatile. If there is a high fraction of word-of-mouth consumers in the population, the 'popularity weighting' property becomes relevant. The volatility of market shares, then, strongly depends on the current market split. E.g., if a firm has a small customer base, it can

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sales. However, the concept of demand inertia received little attention in the literature (in contrast to the concept of subgame perfection), which is presumably due to a lack of a convincing microeconomic foundation.

6 Consumers only remember the information about the supplier where they made their last purchase.

7 See also Stahl (1989).

8 Ellison and Fudenberg (1995) introduce a more general word-of-mouth process where agents can ask a sample of *N* other agents. However, the simpler process assumed here also captures the "popularity weighting property" that characterizes word-of-mouth communication. This means that a firm with a large customer base is in a favorable position, as information about its offer spreads rapidly among consumers.

9 Rob and Fishman (2005) study a market with continuous investments in quality, where word-of-mouth communication means that new consumers who enter the market meet with a certain probability old consumers and find out about the "tenure" of the firm that was patronized by the old consumer last period. The tenure is the number of periods since the firm last produced a low-quality product. However, the authors abstract from price competition, by assuming that each supplier has monopoly power over the consumers visiting it this period.

10 Searchers pass on information to word-of-mouth consumers only about their previous supplier. Results are qualitatively the same if they pass on information about both suppliers. Satterthwaite (1979) analyzes word-of-mouth communication in a market for physicians, where consumers also pass on information they have only heard of from other consumers.

11 Other authors, e.g. Salop and Stiglitz (1977), Varian (1980), Janssen and Moraga-Gonzales (2004), also introduce consumers who learn only one supplier's price, but assume that they choose this supplier randomly. See also Fishman and Rob (1995).

only attract a small number of new customers when it charges the lower price, because few consumers discover this firm via word-of-mouth. Hence, there is a high degree of inertia. However, when the market split is a more even one, many consumers discover both suppliers. Hence, market shares are volatile.

The second aspect that affects the dynamics in this game is strategic interaction. E.g., a firm with a large customer base may have a higher incentive to exploit its customer base, rather than to compete fiercely for new customers. In contrast, a firm with a small customer base may be a more aggressive player. Hence, there is a tendency towards the center of the market share space, where market shares are (roughly) evenly distributed. However, firms also take into account how the current market outcome affects their *future* profitability. A firm may, thus, try to invest in a large customer base to become the "dominant player" in the market. Once a dominant position is reached (high market share), this firm may vigorously defend its position and, thus, play *more* aggressively than its smaller competitor. In this case, a tendency towards the extremes of the market share space emerges.

In order to disentangle these different effects, the analysis of market share dynamics in this paper is divided into four subcases. In the simplest case ("deterministic benchmark"), firms alternate in gaining and loosing market shares. This yields a simple characterization of market share dynamics in the absence of strategic interaction, and reveals dynamic aspects that are "inherent" to the process of information transmission. Using a fixed point analysis, analytical results are derived to describe the patterns of market share evolution. In the second case ("stochastic benchmark"), it is assumed that each firm gains or looses market shares in each period with a fixed probability of 1/2. This yields a more realistic characterization of market share dynamics in the absence of strategic interaction, as it entails a randomization of prices (similarly as in a mixed strategy equilibrium). However, the dependency of current pricing strategies from past market shares is suppressed. As analytical results are difficult to obtain, results are obtained by simulation. The third case under consideration ("repeated static game") considers the evolution of market shares under strategic interaction, when future profits are fully discounted. The incentives to invest in future market shares are, thus, "switched off", which helps to isolate the effects stemming from current profit maximization. Finally, the full dynamic model is analyzed, when future profits are *not* fully discounted. Hence, the incentives to invest in future market shares affect the dynamics.

The main results of the analysis are as follows. The fixed-point analysis of the "deterministic benchmark" case reveals that there are "attractors" in the market share space, to which market shares converge. The location of these attractors depends on the volatility of market shares, and on the prevalence of word-of-mouth communication. Under the "stochastic benchmark", market shares do not converge to the attractors, due to the stochastic nature of the process. However, they tend to be near the attractors with high probability. When most consumers rely on word-of-mouth, the attractors are located near the extremes of the market share space, and a firm that has reached a dominant market position can often maintain this position for many consecutive periods. When most consumers are of the ignorant type, the attractors are located near the center of the market share space. In this case, the market split tends to be an even one most of the time. The analysis of the "repeated static game" reveals that market shares tend to be closer to the center than in the stochastic benchmark, as a firm with a smaller customer base prices more aggressively and, thus, tends to gain market shares. When future profits are *not* fully discounted ("full dynamic game"), the incentives to invest in future market shares also affect the dynamics. As pointed out above, a firm with a dominant market position may now play more aggressively than its smaller competitor, in order to defend the dominant position. This effect turns out to be particularly pronounced when many consumers communicate via word-of-mouth. Situations where the dominant firm exploits its customer base by charging a high price, then, alternate with situations where it vigorously defends its market position and, thus, prices aggressively. Overall, market shares are more *skewed* when future profits are important. This result seems rather novel in the literature on dynamic price competition.

In an extension, situations are considered where consumers do not return to the market in every period, but in regular intervals or with a certain frequency. This may e.g. reflect the degree of durability of the good sold in this market. When the frequency of repeat purchasing is low, there is more inertia in the market shares. This case is e.g. used to simulate entry dynamics into the market. It is shown that, in order to obtain a sizable market share, an entrant must undercut the incumbent's price for an extended period of time.

The remainder of this paper is organized as follows. In Section 2, the model is introduced, and equilibrium conditions are derived. Section 3 analyzes market share dynamics in the absence of strategic interaction. Section 4 characterizes the dynamics under strategic interaction. Section 5 extends the model to situations where consumers do not return to the market in each period. Section 6 concludes.

2. The model

There are two firms ($i \in \{1,2\}$) in a market for a homogeneous good. Time is discrete, the horizon infinite, and firms simultaneously choose prices $p_{i,t}$ in every period t ($t = 1,2,...$). Marginal costs are constant and normalized to zero. On the demand side, there is a continuum of consumers with measure 1. Each consumer purchases either zero or one unit in each period, and all consumers have the same reservation price, equal to 1. Prices above 1 (the monopoly price) can be eliminated from the strategy space without loss of generality. Hence, market demand in each period equals 1. By assumption, market shares in period t depend on prices in period t, and on market shares in period $t-1$. This "inertia" reflects problems of information acquisition. Let the size of firm 1's customer base in period t, n_t, be firm 1's demand (and, thus, market share) in period $t-1$: $n_t \equiv D_{1,t-1}$. The size of firm 2's customer base in period t is $1-n_t$. If consumers possess more information about a firm where they previously made a purchase, a firm with a large customer base is in a favorable position. Suppose, the market share of firm 1 (firm 2) increases from n_t to $h(n_t)$ (from $1-n_t$ to $h(1-n_t)$) if it currently charges the lower price in the market. If it charges the higher price, it drops to $l(n_t)$ (resp. $l(1-n_t)$). If both firms charge identical prices, their market shares remain constant. Hence, firm 1's demand in period t equals (similarly for firm 2):

$$D_{1,t}(p_{1,t}, p_{2,t}, n_t) = \begin{cases} h(n_t) & \text{if } p_{1,t} < p_{2,t} \\ n_t & \text{if } p_{1,t} = p_{2,t} \\ l(n_t) & \text{if } p_{1,t} > p_{2,t} \end{cases} \tag{1}$$

By assumption, the function $h(.)$ is continuous, strictly increasing, and $n < h(n) < 1$ holds $\forall n \in [0,1)$. Hence, a firm that charges the lower price gains market shares, but it does not serve the entire market (unless its market share was equal to 1 in the previous period). The functions $h(.)$ and $l(.)$ must fulfill the following condition:

$$l(n_t) + h(1-n_t) = 1 \quad \forall n_t \in [0,1] \tag{2}$$

This condition states that the aggregate demand equals 1 in each period, for any given market split (captured by the state variable n_t). Note, that for a given specification of $h(.)$ and $l(.)$, the history of market shares up to period t is fully described by the initial size of firm 1's customer base in period 1 (n_1), and a binary sequence $S_t \equiv \{I_\tau\}$ ($\tau = 1,2,...,t$), with $I_\tau = 1$ if $p_{1,\tau} < p_{2,\tau}$ (firm 1 gains market shares in period τ), and $I_\tau = 0$ otherwise.[12]

A specification of h(.) and l(.):

The model and the solution procedure introduced in this Section are quite general and do not require a specification of the functions $h(.)$ and $l(.)$. However, in order to characterize market share dynamics (see Sections 3-5), a specification is needed. Let us, thus, introduce a specification. As outlined in the Introduction, suppose there are three consumer types. The "searchers" have zero search costs and can, thus, always compare prices. A "word-of-mouth consumer" can communicate with one other consumer for free, but faces high search costs. If the consumer he/she asks visited a different supplier in the previous period, this supplier is discovered. Otherwise, the word-of-mouth consumer remains locked-in at his/her own previous supplier. Consumers of the "ignorant type" are always locked-in at their previous supplier. Let λ be the fraction of word-of-mouth consumers in the population, and ρ be the fraction of ignorant types. The fraction of searchers is $1-\lambda-\rho$. By assumption, λ and ρ are independent of the history of the game. Hence, in the customer base of each firm, the same fraction of searchers and word-of-mouth consumers exists in each period. The 'type' of a consumer may, thus, be seen as a transient attribute, that is chosen from the same distribution again in each period.[13] Under these assumptions, we obtain for the function $l(.)$:

$$l(n) = \lambda n^2 + \rho n \tag{3}$$

The function $h(.)$, then, follows from (2): $h(n) = 1 - \lambda(1-n)^2 - \rho(1-n)$.

Profit maximization:

Firm 1's expected profit in the current period is $\pi_{1,t}^E(n_t, p_{1,t}) = p_{1,t} E_{p_{2,t}}\left[D_{1,t}(p_{1,t}, p_{2,t}, n_t)\right]$. Firm 2's profit is obtained by replacing n by $1-n$, and exchanging the indices of the firms. Let δ be the discount factor for future profits. In period t, firm i maximizes the present discounted value of future expected profits over an infinite horizon:

$$V_i(H_t) \equiv \max_{\{p_{i,t}\}} \sum_{\tau=t}^{\infty} \delta^{\tau-t} \pi_{i,\tau}^E(p_{i,\tau} \mid H_\tau) \tag{4}$$

[12] The event $p_{1,\tau} = p_{2,\tau}$ occurs with probability zero in equilibrium and can be neglected. See below.

[13] An alternative interpretation is that each period, new consumers arrive who inherit information from their parents, but not their type.

, where H_t denotes the history up to period t. The history H_t contains all prices chosen up to period $t-1$, as well as the initial state n_1.[14] The expectation in (4) is over firm j's prices ($j \neq i$). Note, that there is no external source of uncertainty in the model. In equilibrium, uncertainty arises only if firms use mixed pricing strategies.

In general, firms can condition their choice in period t on the entire history up to period t, H_t. However, the set of equilibria in dynamic pricing games is potentially overwhelming. In order to narrow the set of equilibria, the Markov perfection equilibrium refinement is used. The payoff-relevant state variable in period t is n_t. Under Markov perfection, firms are restricted to condition their price only on the current state.

The Bellman-equation for a firm with a customer base of size n (firm 1) reads:

$$V(n) = \max_p \left[\pi^E(n,p) + \delta E\left[V(n') \mid n, p \right] \right] \tag{5}$$

, where n and n' stand for n_t and n_{t+1}, respectively. By omitting the subscript for the identity of the firm, (5) contains the following symmetry assumption: a firm's present value only depends on the size of its current customer base, n, and not on the identity of the firm.

Proposition 1: If δ is sufficiently small[15], there is no Markov perfect equilibrium that comprises pure strategies for any $n \in [0,1]$. Hence, firms use mixed pricing strategies. Both equilibrium price distribution functions $F(n,p)$ and $F(1-n,p)$ [16] have the same support (for any given n), the convex set of prices p from $\underline{p}(n) < 1$ (lowest price with positive density) to 1. At most one firm attaches positive probability mass to any single price, and if so, the mass point is located at 1 (monopoly price). (Proof: See the Appendix)

Firm 1's expected profit in the current period is given by:

$$\pi^E(n,p) = p\left(l(n)F(1-n,p) + h(n)\left(1 - F(1-n,p)\right)\right) \tag{6}$$

[14] The states in periods 2 through t, captured by the sequence S_t that describes firm 1's market share gains and losses, can be inferred from the sequence of prices and n_1.

[15] δ must be sufficiently small because an increase in the size of a firm's customer base may lead to *reduced* profits in the future when the intensity of future price competition increases. Hence, the value function $V(n)$ is not generally monotone. If δ is large, a firm may be reluctant to undercut the competitor's price if this price can be predicted with certainty. However, the logic of the mixed pricing strategies is that, within the support of $F(.)$, a firm would always undercut the competitor's price if it were known.

[16] Let the distribution function be defined as: $F(n,p) \equiv \Pr[P < p \mid n]$. Under this convention, $1 - F(n,1)$ is the probability mass at the monopoly price.

, where $l(n)$ is firm 1's demand if it chooses the higher price, and $F(1-n,p)$ is the probability of this event. Firm 1's expected value in the next period is:

$$E\left[V(n')\,|\,n,p\right]=V\left(l(n)\right)F(1-n,p)+V\left(h(n)\right)\left(1-F(1-n,p)\right) \tag{7}$$

, where $V\left(l(n)\right)$ is the firm's value after losing market shares.

In a mixed strategy equilibrium, the maximum in (5) is attained over a range of prices, and the support of $F(n,p)$ contains only prices within this set. For all prices p outside the support of $F(n,p)$, it must hold that: $\pi^E(n,p)+\delta E\left[V(n')\,|\,n,p\right]\le V(n)$. Hence, there is no profitable deviation from the equilibrium randomization strategy $F(n,p)$.

Using (6) and (7) in (5), we obtain the following expression for the value function $V(n)$:

$$V(n)=ph(n)-p\left(h(n)-l(n)\right)F(1-n,p)+\delta V\left(h(n)\right)-\delta\left(V\left(h(n)\right)-V\left(l(n)\right)\right)F(1-n,p) \tag{8}$$

The max-operator has been omitted because, for prices within the support of $F(.)$, the right-hand side must be independent of p. (8) can be solved for $F(1-n,p)$. Replacing $1-n$ by n, we obtain for the distribution function $F(n,p)$:

$$F(n,p)=\frac{\delta V\left(h(1-n)\right)-V\left(1-n\right)+h(1-n)p}{\delta V\left(h(1-n)\right)-\delta V\left(l(1-n)\right)+\left(h(1-n)-l(1-n)\right)p}\;,\;\forall n\in[0,1] \tag{9}$$

If $V(.)$ is known, (9) allows to compute the distribution function $F(.)$ for all n.

Equilibrium conditions:

In the following, equilibrium conditions are derived that allow to determine $V(.)$. By Proposition 1, it must hold $\forall n\in[0,1]$ that: $F(n,\underline{p})=F(1-n,\underline{p})=0$. Using (9), we, thus, obtain the first equilibrium condition:

$$h(n)V\left(1-n\right)-\delta h(n)V\left(h(1-n)\right)=h(1-n)V\left(n\right)-\delta h(1-n)V\left(h(n)\right) \tag{10}$$

The other conditions follow from the result that at most one of the firms chooses the monopoly price with positive probability (see Proposition 1). Hence, it must either hold that $F(n,1)\le 1$ and $F(1-n,1)=1$ ("case a"), or that $F(n,1)=1$ and $F(1-n,1)<1$ ("case b"). Suppose, for a given value of the state n, case a is relevant.[17] $F(1-n,1)=1$ can, thus, be used to derive an equilibrium condition. Using (9), we obtain:

$$V\left(n\right)-\delta V\left(l(n)\right)=l(n) \tag{11}$$

[17] Until now, it is not clear under what conditions case a or case b is relevant. Intuitively, we would expect that case a is relevant (firm 1 chooses the monopoly price with positive probability) whenever firm 1's customer base n is greater than 1/2, because it may, then, have a lower incentive to compete for new customers, while firm 2 may play more aggressively. This intuition is generally correct, but some complications can arise (see below).

If case b is relevant (given n), use $F(n,1)=1$ in (9) to obtain the following condition:

$$V(1-n)-\delta V(l(1-n))=l(1-n) \tag{12}$$

Together with an (until now) unknown rule that states whether case a or case b is relevant for every given state n, (10), (11), and (12) implicitly define the value function $V(.)$. Note, that this is a continuum of equations because these conditions must be fulfilled $\forall n \in [0,1]$.

Solution procedures:

There are (at least) two possible solution procedures. One is based on a *discretization* of the state space ($n \in [0,1]$). Conditions (10) and (11), resp. (12), can, then, be solved analytically with the help of a computer. Note, that this is simply a system of linear equations. The only difficulty is that the rule that determines when case a and case b is relevant (that is, whether (11) or (12) must be used for a given n) is not known. The most obvious guess is that case a is relevant if $n \geq 1/2$, and case b otherwise. Using this guess, a candidate equilibrium can be computed. It must, then, be verified that $F(1-n,1)<1$ holds for all $n \in [0,0.5)$, and $F(n,1) \leq 1$ for $n \in [0.5,1]$. Otherwise, the decision rule must have been wrong, or no equilibrium exists for the given parameter values.[18]

Another possibility is to *iterate* on the value function. While the first procedure yields an analytical solution to an approximated problem, the iteration yields an approximate solution to the original problem. This procedure has the advantage that a decision rule that states whether case a or case b is relevant for a given n, can be derived analytically. The iteration procedure is now described in more detail. Let $\tilde{V}$ be the iterated value function. $\tilde{V}$ is computed recursively. To start the iteration, simply set $\tilde{V}(n) \equiv 0 \;\; \forall n \in [0,1]$. The equilibrium conditions (10), (11), and (12) yield (after rearranging): [19]

$$\tilde{V}(n)=\frac{h(n)}{h(1-n)}\left(\tilde{V}(1-n)-\delta\tilde{V}'(h(1-n))\right)+\delta\tilde{V}'(h(n)) \tag{13}$$

$$\tilde{V}(n)=l(n)+\delta\tilde{V}'(l(n)) \tag{14}$$

$$\tilde{V}(1-n)=l(1-n)+\delta\tilde{V}'(l(1-n)) \tag{15}$$

[18] It turns out that, if δ is sufficiently small, the above decision rule is always correct, and the solution procedure yields good results if the market share grid is sufficiently fine. Problems arise only if the parameters δ and λ are both fairly large. The difficulties are related to feedback effects stemming from the popularity weighting property of the word-of-mouth process.

[19] Taking account of the iteration procedure, the value function of the current iteration, $\tilde{V}$, and the value function of the previous round, $\tilde{V}'$, must be distinguished (they are identical when the iteration has converged).

(14) is valid if case a is relevant. If this holds for the given value of n, then it must also hold that $F(n,1) \le 1$. Using (13) and (14) in (9), this yields the following condition:

$$\frac{h(1-n) - \frac{h(1-n)}{h(n)}\left(l(n) + \delta\tilde{V}'\left(l(n)\right) - \delta\tilde{V}'\left(h(n)\right)\right)}{\delta\tilde{V}'\left(h(1-n)\right) - \delta\tilde{V}'\left(l(1-n)\right) + \left(h(1-n) - l(1-n)\right)} \le 1 \tag{16}$$

If (16) is fulfilled, case a is relevant, if it is violated, case b is relevant. Using the definition:

$$\beta(n) \equiv h(1-n)\left(l(n) + \delta\tilde{V}'\left(l(n)\right) - \delta\tilde{V}'\left(h(n)\right)\right) \tag{17}$$

, (16) can be written more conveniently as follows:

$$\beta(n) \ge \beta(1-n) \tag{18}$$

(18) holds with equality for $n = 1/2$. Therefore, at this point, there is always a (potential) switch from case a to case b (or vice versa). For all values of n that fulfill (18), the iteration formula (14) can be used in the current iteration round. For all other values of n, case b is relevant. Plugging the expression for $\tilde{V}\left(1-n\right)$ in (15) into (13), we obtain the following iteration formula for case b (using (17)):

$$\tilde{V}\left(n\right) = \frac{\beta(1-n)}{h(1-n)} + \delta\tilde{V}'\left(h(n)\right) \tag{19}$$

Combining (14), (18), and (19), we obtain the following iteration formula for all values of n:

$$\tilde{V}(n) = \begin{cases} l(n) + \delta\tilde{V}'\left(l(n)\right) & \text{if } \beta(n) \ge \beta(1-n) \\ \dfrac{\beta(1-n)}{h(1-n)} + \delta\tilde{V}'\left(h(n)\right) & \text{if } \beta(n) < \beta(1-n) \end{cases} \tag{20}$$

Using (20), the iteration procedure is easy to implement on a computer. For small values of δ, it converges quickly. If λ and δ are both fairly large, the iteration process often takes many rounds to converge or fails to converge. As a general rule, the iteration process converges whenever (18) is fulfilled with equality *only* at $n = 1/2$ (and no other value of n) after several iteration rounds, which corresponds to the postulated simple decision rule mentioned above. The iteration process, and the procedure with the discretized state space, then, yield identical results up to some numerical imprecision. Apparently, the lack of a solution (if it occurs) is related to highly irregular patterns in the shape of the invariant distributions (see Section 4). It turns out, that a small change in the initial state n_1 may, then, affect the probability distribution of the firms' market shares over many future periods and can, thus, lead to a substantial change in a firm's value. Hence, the value function may not be continuous. In such situations, the dynamic program may not have a solution.[20]

3. Market share dynamics in the absence of strategic interaction

Before analyzing market share dynamics generated under strategic interaction, it is useful to analyze the dynamics "inherent" to the process of information transmission (introduced in Section 2) in a *non-strategic* environment. Section 3.1 analyzes the "deterministic benchmark", where firms simply alternate in gaining and losing market shares. It is shown that there are "attractors" in the market share space, to which the market shares converge. Section 3.2 considers the "stochastic benchmark", where in each period, each firm gains or looses market shares with an equal probability. This is more similar to the mixed-pricing equilibria under strategic interaction, but the history-dependence of current pricing strategies is "switched off". Due to the stochastic environment, market shares do not converge to the attractors, but they tend to be close to them with high probability.

3.1 Deterministic benchmark

Consider the evolution of *firm 1*'s market share under the "alternating sequence" $S^{alt} \equiv (0,1,0,1,...)$: in all even- (odd-) numbered periods, firm 1 gains (loses) market shares.[21]

Definition: n^{fix} is a *fix point* in the market share space if it fulfills: $n^{fix} = h\left(l(n^{fix})\right)$.

Hence, if firm 1's market share is initially n^{fix}, and the firm first loses and then gains market shares, it reaches its original market share n^{fix} again. The fix point is, thus, reached in all *even-numbered* periods under the alternating sequence S^{alt}.[22] To simplify the exposition, it is useful to define: $g(n) \equiv h\left(l(n)\right)$. Under the alternating process, the sequence of firm 1's market shares in even-numbered periods is, thus, given by the following law of motion: $n_t = g(n_{t-2})$, and any fix point must fulfill: $n^{fix} = g(n^{fix})$.

Lemma 1: If n^{fix} is a fix point, then $h(1-n^{fix})$ is also a fix point ("*corresponding fix point*").

---

[20] The observation that, for some parameter values, a small change in the starting value n_1 can lead to a markedly different evolution of market shares over many periods is reminiscent of chaotic behavior.

[21] Remember, that firm 2's market share in period t is simply $1-n_t$.

[22] In all odd-numbered periods, the market share $l(n^{fix})$ is attained.

Proof: $h(1-n^{fix})$ is a fix point if $h(1-n^{fix}) = g(h(1-n^{fix})) = h\left(l\left(h(1-n^{fix})\right)\right)$ holds. By strict monotonicity of $h(.)$, this is equivalent to: $1-n^{fix} = l\left(h(1-n^{fix})\right)$. Using (2), this can be written as: $n^{fix} = h\left(l\left(n^{fix}\right)\right) = g(n^{fix})$, which is true since n^{fix} is a fix point. □

Definition: n^{fix} is *degenerate* if the corresponding fix point $h(1-n^{fix})$ is identical to n^{fix}.

To find out whether a fix point $n^{fix} > 1/2$ is "degenerate", simply compute $l(n^{fix})$. If this is below 1/2, then the fix point is degenerate, which means that firm 1's market shares under S^{alt} alternate *symmetrically* around 1/2. If $l(n^{fix}) > 1/2$, then n^{fix} is non-degenerate, and (by symmetry) there exists a corresponding fix point in the lower half of the market share space, located at $h(1-n^{fix})$. Given an arbitrary starting point, a sequence of market shares can either converge to a fix point, or diverge away from it.

Definition: n^{fix} is an *attractor* (stable fix point) if there is a neighborhood $U_\varepsilon(n^{fix}) \equiv \left\{n \in [0,1]: \left|n-n^{fix}\right| < \varepsilon\right\}$ with $\varepsilon > 0$, such that every sequence of firm 1's market shares that starts from some n_1 within $U_\varepsilon(n^{fix})$, converges to n^{fix} under $S^{alt} = (0,1,0,1,...)$.

Lemma 2: n^{fix} is an attractor if $\left|dg(n)/dn\right|_{n=n^{fix}} < 1$.[23]

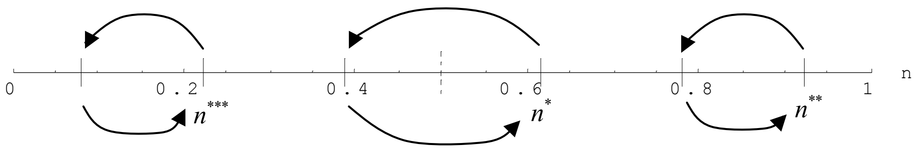
For the process of information transmission introduced in Section 2, we obtain the following solutions to the condition $n = g(n)$, for the interval $n \in [0.5,1]$ (using (3)): [24]

$$n^* \equiv \left(\sqrt{(1+\rho)^2 + 4\lambda} - 1 - \rho\right)/2\lambda \ , \ n^{**} \equiv \left(\sqrt{(1+\rho)^2 - 4(1-\lambda)} + 1 - \rho\right)/2\lambda \tag{21}$$

n^* is a degenerate fix point, and n^{**} a non-degenerate one. Hence, another fix point is located at $n^{***} \equiv h(1-n^{**}) < 0.5$. This is illustrated in Figure 1 (for $\lambda = 0.7$ and $\rho = 0.2$):

[23] This is a standard result for dynamic systems. A formal proof is, thus, omitted.
[24] There is another solution that is located outside the interval [0,1].

Figure 1: Fix points in the market share space; $\lambda = 0.7$, $\rho = 0.2$



In the following, two special cases are analyzed: 1. $\rho = 0$ (there are no consumers of the ignorant type), and 2. $\lambda = 0$ (there are no word-of-mouth consumers).[25]

Special case: $\rho = 0$

Using $\rho = 0$, we obtain: $g(n) = 1 - \lambda\left(1 - \lambda n^2\right)^2$. The expressions for n^* and n^{**} in (21) simplify to: $n^* \equiv \left(\sqrt{4\lambda + 1} - 1\right)/2\lambda$ and $n^{**} \equiv \left(\sqrt{4\lambda - 3} + 1\right)/2\lambda$. Figure 2 shows n^* and n^{**}.

Figure 2: Fix points n^* and n^{**} as a function of λ; $\rho = 0$ (dashed: unstable fix point)

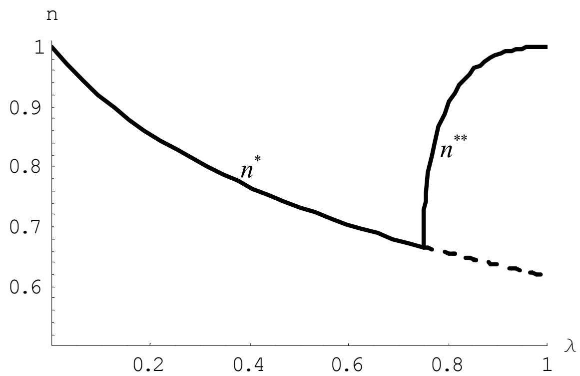


Figure 2 illustrates that n^{**} exists only for $\lambda \geq 3/4$, while n^* exists for $\lambda \in [0,1]$.

Proposition 2: For $\rho = 0$, n^* is an attractor if $\lambda < 3/4$, and n^{**} is an attractor if $\lambda > 3/4$.

Proof: By Lemma 1, stability requires that $\left|dg(n)/dn\right|_{n=n^{fix}} < 1$. Using $g(n) = 1 - \lambda\left(1 - \lambda n^2\right)^2$, $n^* \equiv \left(\sqrt{4\lambda + 1} - 1\right)/2\lambda$, and $n^{**} \equiv \left(\sqrt{4\lambda - 3} + 1\right)/2\lambda$, this yields $\lambda < 3/4$ (resp. $\lambda > 3/4$). □

Hence, for $\lambda < 3/4$ (and $\rho = 0$), firm 1's market share converges to n^* under S^{alt} for any starting point n_1. For $\lambda > 3/4$, it converges to n^{**} if $n_1 > n^*$, and to n^{***} if $n_1 < n^*$.

---

[25] This distinction is made for tractability.

In the following, a simple measure of the *volatility* of market shares is introduced. Let k be the minimum number of consecutive market share losses of firm 1, that (starting from $n_1 = 1$) are sufficient to yield a market share of 1/2 (and hence, destroy firm 1's dominant position).[26] To compute k, apply $l(n) = \lambda n^2$ k times to $n_1 = 1$ to get: $l^k(1) = \lambda^{2^k - 1}$. Equalize this to 1/2, and solve for k to obtain:

$$k = \ln\left(1 - \frac{\ln 2}{\ln \lambda}\right) / \ln 2 \tag{22}$$

Since k is usually not an integer, the minimum number of consecutive market share losses until firm 1's market share falls below 1/2, is given by the smallest integer greater than k. For example, for $\lambda = 0.91$, we have $k \cong 3$. Hence, for all $\lambda > 0.91$, at least four consecutive market share losses are required to destroy firm 1's dominant position in the market.

Special case: $\lambda = 0$

For $\lambda = 0$, we get: $g(n) = 1 - \rho(1 - \rho n)$. The expression for n^* in (21) simplifies to: $n^* = 1/(1+\rho)$. n^{**} does not exist for $\lambda = 0$.

Proposition 3: For $\lambda = 0$, n^* is always an attractor (for any $\rho < 1$).

Proof: $|dg(n)/dn| = \rho^2 < 1$. □

For $\lambda = 0$, the minimum number of consecutive market share losses (starting from $n_1 = 1$), such that firm 1's market share falls below 1/2, is given by the smallest integer greater than $k = -\ln 2 / \ln \rho$.[27] For example, $k(0.79) \cong 3$. Hence, if $\rho > 0.79$, at least four consecutive market share losses are required to destroy firm 1's dominant position (starting from $n_1 = 1$). Note, that this does *not* imply that market shares will be near the extremes with a high probability under the stochastic benchmark, because for large values of ρ, the attractor n^* is close to 1/2. It only implies that, when a firm has a large (or a small) market share initially, a more even split of the market is reached only gradually.

---

[26] Under the stochastic benchmark, we will see that only if k is large, *and* the information process favors skewed distributions of market shares (a stable fix point is located near $n = 1$), then a firm with a dominant position is likely to maintain this position for many consecutive periods. If there is a stable fix point near $n = 1$, but k is small (this is the case when λ and ρ are both small), market shares tend to fluctuate between the extremes, and usually do *not* stay close to one extreme (n=0, n=1) for many consecutive periods.

[27] To see this, apply $l(n) = \rho n$ k times to $n_1 = 1$ to obtain: $l^k(1) = \rho^k$. Equalize this to 1/2 and solve for k.

3.2 Stochastic benchmark

We are now equipped with some qualitative predictions about the market share dynamics in the absence of strategic interaction. E.g., when λ and ρ are both small, the attractor n^* is located close to 1, and the volatility of market shares is high (k is small). Hence, market shares tend to fluctuate between the extremes (between values close to $n = 0$ and close to $n = 1$), but usually do not stay close to one extreme for many consecutive periods. When $\rho = 0$ and λ is large (close to 1), the stable attractor n^{**} is located close to 1, and the volatility of market shares is low (k is high). Hence, the market split tends to be skewed most of the time, and once a firm reaches a dominant position in the market, it is likely to maintain this position for many consecutive periods. This is due to the popularity weighting property of word-of-mouth. If $\lambda = 0$ and ρ is large, the attractor n^* is close to 1/2, and the volatility of market shares is low. Hence, the market split tends to be an even one most of the time.

In the following, these predictions are confirmed by means of simulation. Analytical results are difficult to obtain even for the simple stochastic benchmark, because – as will become clear below – the invariant distributions of market shares are often highly irregular.

Figure 3 shows a simulation of market shares under the stochastic benchmark, for $\lambda = 0.95$ and $\rho = 0$, and a numerical approximation of the invariant distribution for these parameter values.[28] Note, that the invariant distribution shows the probability of each possible state, but it does not reveal the patterns of transition between the states.

Figure 3: Simulated evolution (left) and invariant distribution (right) of market shares, stochastic benchmark; $\lambda = 0.95$, $\rho = 0$

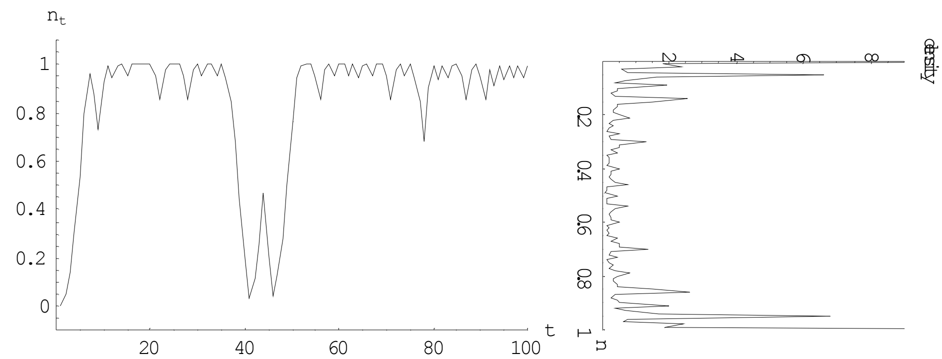


Figure 3 confirms the prediction that, for large values of λ, the market split tends to be skewed most of the time, and once a firm reaches a dominant position in the market, it is likely to maintain this position for many consecutive periods. The invariant distribution

[28] The invariant distributions are also computed by simulation, but with a much higher number of periods (usually between 10^5 and 10^6). For the market share grid, a uniform step size of 0.01 was used.

shown in Figure 3 reveals a highly irregular shape. Market shares tend to be skewed with a high probability, but there are "hot spots" (peaks) to which market shares return more often than to other points located close to them. Furthermore, the location of these peaks is sensitive to the exact parameter values. A small change in λ can lead to a different shape of the invariant distribution (not shown). This gives an idea why analytical results are virtually impossible to derive even for the (conceptually) simple stochastic benchmark.
Figure 4 shows simulated market shares and the invariant distribution for $\lambda = 0$ and $\rho = 0.95$.

Figure 4: Simulated evolution and invariant distribution of market shares, stochastic benchmark; $\lambda = 0$, $\rho = 0.95$

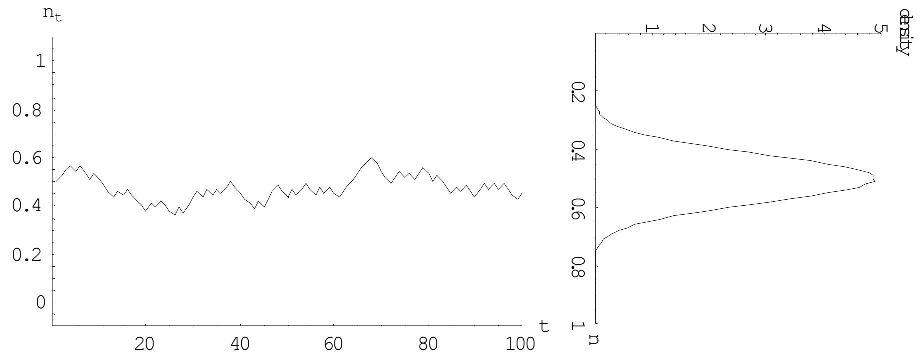


Figure 4 confirms the prediction that for high values of ρ, the market split tends to be an even one most of the time. Note, that the invariant distribution now reveals a regular shape. However, this is not generally true for the case $\lambda = 0$. For smaller values of ρ, the invariant distribution becomes highly irregular again (not shown). The comparison of Figure 4 with Figure 3 illustrates, that the prevalence of word-of-mouth communication crucially affects the dynamics of market shares.
An interesting special case is the one where exactly half of the consumers are of the ignorant type, and the other half are searchers ($\lambda = 0$, $\rho = 0.5$). This case is simulated in Figure 5.

Figure 5: Simulated evolution and invariant distribution of market shares, stochastic benchmark; $\lambda = 0$, $\rho = 0.5$

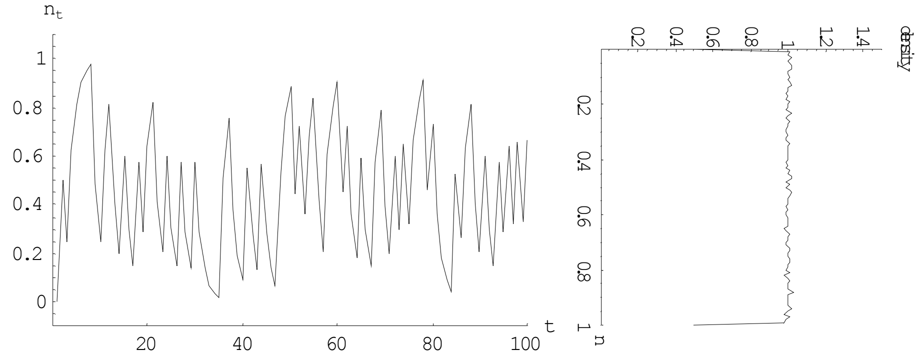


Figure 5 reveals that, for these particular parameter values, the invariant distribution is uniform.[29] Hence, all market shares in the interval (0,1) occur with the same probability. Note, however, that when ρ differs only marginally from 1/2, the shape of the invariant distribution becomes highly irregular again (not shown).

The analysis in Section 3.1 focused on two special cases, namely the case where $\rho = 0$, and the one where $\lambda = 0$. Another interesting case to consider is a situation where most consumers either communicate via word-of-mouth, or are of the ignorant type. To simulate this case, a small but positive fraction of searchers must be maintained, for otherwise, the market share of a firm will at some point converge either to 0 or to 1. This is because, when there are no searchers, nobody finds out about a supplier with a market share of zero. Figure 6 shows a simulation of market shares and the invariant distribution for $\lambda = \rho = 0.49$.

Figure 6: Simulated evolution and invariant distribution of market shares, stochastic benchmark; $\lambda = \rho = 0.49$

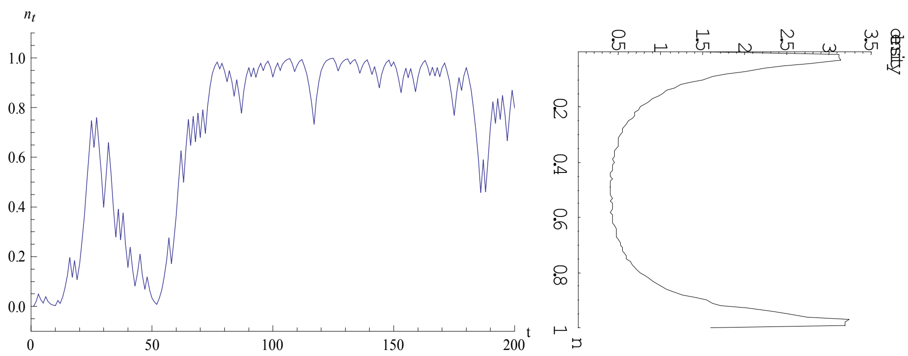


Figure 6 reveals that properties inherent to the word-of-mouth process (namely skewed market shares) "overlap" with properties that are typical for a situation with many ignorant consumers. Overall, market shares are not very volatile, and intervals where they are skewed alternate with (shorter) intervals where the market split is roughly an even one.

4. Market share dynamics under strategic interaction

Section 2 showed that firms use mixed pricing strategies in equilibrium. The resulting price dispersion generates market share dynamics. These dynamics are characterized in this Section. Two cases are analyzed separately: the "repeated static game" (Section 5.1), and the full dynamic game (Section 5.2). While in the latter, the incentives to invest in future market shares play an important role, they are excluded in the repeated static game. Market share

[29] The small "bumps" are due to the finite number of simulation rounds.

dynamics in the full dynamic game are compared to those under the repeated static game to reveal the impact of these incentives. The repeated static game is compared to the stochastic benchmark (Section 3.2) to reveal the impact of current profit maximization on the market share dynamics. Table 1 summarizes the four cases analyzed in this paper.

Table 1: Four cases analyzed in this paper

case:	reveals:
1. deterministic benchmark	- location of attractors in market share space
2. stochastic benchmark	- dynamic properties "inherent" to the process of information transmission
3. repeated static game	- dynamics resulting from current profit maximization
4. full dynamic game	- incentives to invest in future market shares

4.1 Repeated static game

As in Section 3.2, the evolution of market shares is characterized by means of simulation.[30] Let us start with a situation where most consumers are searchers (λ and ρ close to 0). This case is interesting, as it is only a small step away from the standard Bertrand model with fully informed consumers (with trivial market share dynamics). Figure 7 shows a simulation of market shares and the invariant distribution for $\lambda = 0.1$ and $\rho = 0$.

Figure 7: Simulated evolution and invariant distribution of market shares, repeated static game ($\delta = 0$); $\lambda = 0.1$, $\rho = 0$

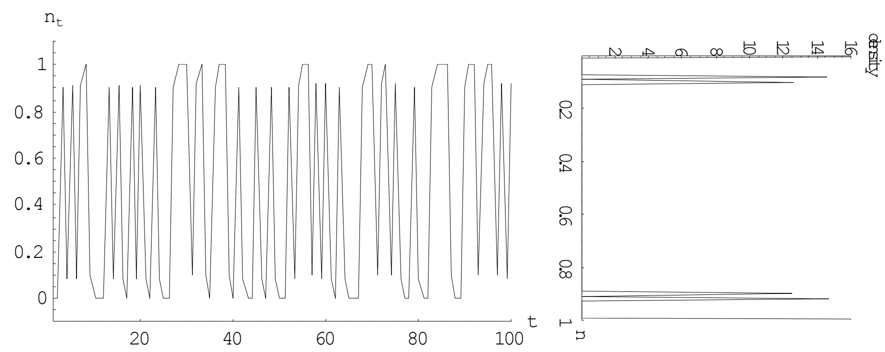


Figure 7 illustrates that for λ and ρ near 0, market shares are very volatile and fluctuate between the extremes of the market share space. This behavior of market shares does not

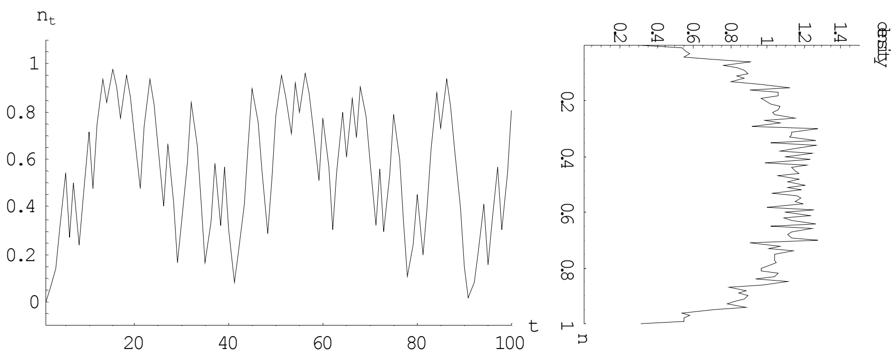
[30] For the equilibrium pricing strategies in the repeated static game, a closed-form solution can be obtained. To this end, use $\delta = 0$ in (9), (17), and (20) (not shown).

seem to be a plausible prediction for most markets in the real world, where market shares generally appear to be less volatile. Hence, consumers are either less well-informed in the real world, or other mechanisms (e.g. product differentiation, superior production technologies, innovations...) create additional inertia in the market shares. The invariant distribution reveals that there are essentially four different splits of the market that occur with positive probability. One of these splits (consisting of two peaks) is located near $n = 0.9$. This split is reached if the size of firm 1's customer base is initially close to 0, and firm 1 gains market shares in the current period. Another "hot spot" is located close to 1. This spot is reached if n is initially around 0.9, and firm 1 once more gains market shares. The other two "hot spots" are located in the lower half of the market share space (their location follows from symmetry).

These findings can be compared with the stochastic benchmark. It turns out that for λ and ρ near zero, the dynamics in the stochastic benchmark are very similar to those under the repeated static game. This is related to the fact that in both cases, the probability of gaining or losing market shares is near 1/2, irrespective of the initial split of the market in period t (captured by the state n_t). However, in the repeated static game, the probability of maintaining a dominant market position for several consecutive periods is slightly lower, because a firm with a small customer base prices more aggressively than its competitor. Hence, this firm gains market shares with a probability (slightly) above 1/2.

Figure 8 simulates a market where most consumers communicate via word-of-mouth. [31]

Figure 8: Simulated evolution and invariant distribution of market shares, repeated static game; $\lambda = 0.95$, $\rho = 0$



The comparison of the dynamics in Figure 8 with those under the stochastic benchmark in Figure 3 (that were simulated for the same parameter values), illustrates that market shares in the repeated static game tend to be *less skewed*. Furthermore, a firm that has reached a dominant market position, is *less* likely to maintain this position for many consecutive periods

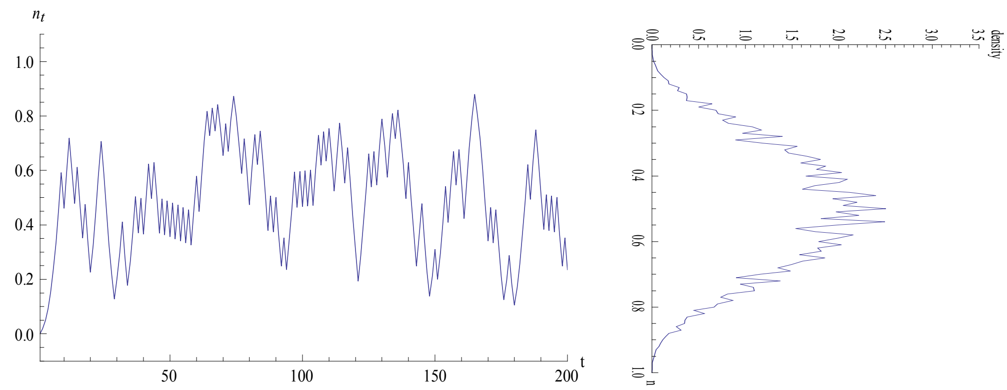
[31] Note, that the irregular shape of the invariant distribution in Figure 8 is *not* due to numerical imprecision.

than under the stochastic benchmark. The reason for this is, that a firm with a small customer base has a stronger incentive to undercut the competitor's price in order to attract additional customers, while the other firm tends to exploit the locked-in consumers in its customer base. Hence, the firm with the smaller customer base tends to gain market shares, which yields a tendency towards the center of the market share space. This effect partly neutralizes the tendency inherent to the information process towards skewed market shares (due to the popularity weighting property of word-of-mouth), but does not entirely eliminate it.

The dynamics of the repeated static game in situations where many consumers are of the ignorant type (ρ close to 1), are qualitatively similar to those under the stochastic benchmark (not shown). However, as in the case where λ is large, there is a stronger tendency towards even splits of the market, as a firm with a small customer base prices more aggressively and is, thus, likely to gain market shares.

The effect is more pronounced in a situation where most consumers either communicate via word-of-mouth, or are of the ignorant type. Figure 9 shows a simulation for $\lambda = \rho = 0.49$.

Figure 9: Simulated evolution and invariant distribution of market shares, repeated static game; $\lambda = \rho = 0.49$



The comparison of Figure 9 with Figure 6 (simulated for the same parameters) shows clearly the tendency towards more even splits of the market under the repeated static game than under the stochastic benchmark.

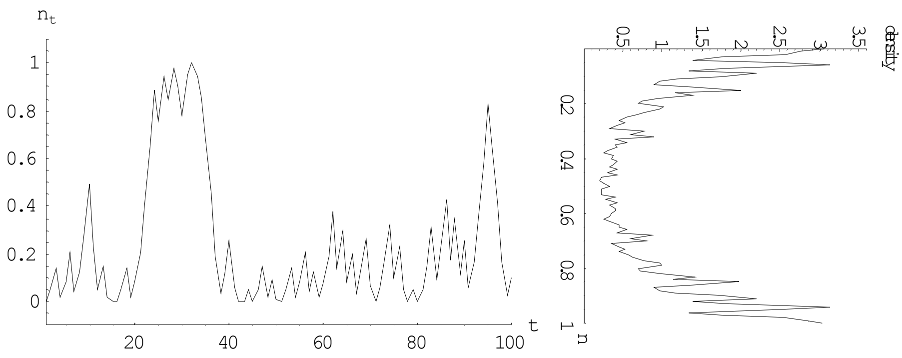
4.2 Full dynamic game

The evolution of market shares for $\delta > 0$ is analyzed using simulations based on the approximated value function $\tilde{V}(n)$. The value function is approximated using the iteration procedure or the method of discretizing the state space (see Section 2).[32] The first result is,

[32] The procedure with the discretized state space is computationally faster when the iteration takes many rounds to converge. Both methods yield (up to some numerical imprecision) identical results. Note, that market share

that for values of λ and ρ near zero, the dynamics for $\delta > 0$ are very similar to those under the repeated static game (see Figure 7). This is because, when most consumers are fully informed, the firm that currently offers the lower price serves most of the market, *irrespective* of the size of its customer base. Hence, firms have little incentive to invest in market shares. Figure 10 simulates a market where most consumers communicate via word-of-mouth.

Figure 10: Simulated evolution and invariant distribution of market shares, full dynamic game; $\lambda = 0.95$, $\rho = 0$, $\delta = 0.5$



The comparison of Figure 10 with Figure 8 (simulated for the same parameters except δ) illustrates, that for larger values of the discount factor δ, market shares tend to be more *skewed*, and a firm that has obtained a dominant position is more likely to maintain this position for several consecutive periods. Hence, the evolution of market shares is qualitatively similar to the stochastic benchmark. This can be explained as follows. When future profits are important, a firm with a large customer base has an incentive to *defend* its dominant position in the market. This leads to more intense price competition near the extremes, which partly neutralizes the tendency towards more even market splits observed for $\delta = 0$. This effect is particularly strong when market shares are not *at* the extremes, but relatively close to them (say, $n \cong 0.2$ or 0.8), as can be confirmed by analyzing the location of the price distribution functions. For $n = 1$, firm 1's distribution function f.o.s. dominates firm 2's, so firm 1 is likely to lose market shares, as it prefers to exploit the locked-in consumers in its customer base. However, for slightly lower values of n, the situation is reversed. Firm 1 is now likely to gain market shares again, as it starts to vigorously defend its dominant market position. Therefore, market shares tend to fluctuate near the extremes (compare Figure 10 also with Figure 3).

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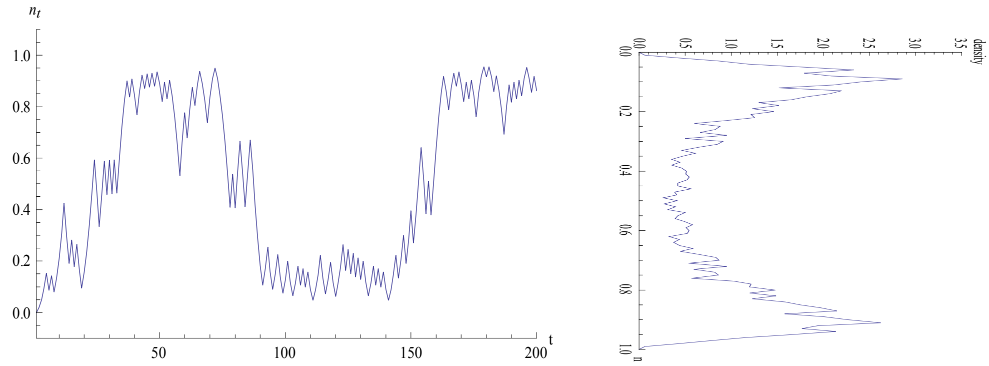
dynamics are closely linked to the shape of the value function, and can not be understood in isolation. The link is as follows. The value function implies the location of the distribution functions (via (9)) for every possible state n. The distribution functions generate the process of market share evolution. The value function itself, however, depends on the stochastic properties of this process.

Similar results are obtained for other parameter values. E.g., when many consumers are of the ignorant type and future profits are fully discounted, the market split tends to be an even one most of the time when. When δ increases, market shares tend to be more skewed. However, the effect is less pronounced than in a situation with many word-of-mouth consumers.

Figure 11 shows a simulation for a situation where most consumers either communicate via word-of-mouth, or are of the ignorant type ($\lambda = \rho = 0.49$).

Figure 11: Simulated evolution and invariant distribution of market shares, full dynamic game; $\lambda = \rho = 0.49$, $\delta = 0.75$



The comparison of Figure 11 with Figure 9 (same parameters except δ) clearly shows the tendency towards *skewed* market splits when future profits are *not* fully discounted (the dynamics are similar to those under the stochastic benchmark, Figure 6).

The main result of this Section, namely that more skewed market splits emerge when future profits are important, is surprising. Intuitively, one may expect that a firm with a small customer base prices aggressively to gain new customers. Hence, there would be a tendency towards even splits of a market (as shown in Section 4.1 for the repeated static game). However, in a fully dynamic environment where future profits are important, an even split of the market may no longer be the most "natural" outcome. There can be extended periods of time where a firm dominates the market, as it vigorously defends its dominant position whenever it is threatened. The smaller competitor can "steel" the dominant position by pricing aggressively for many periods, but this may be too costly. Hence, asymmetric splits of a market can persist for extended periods of time, even when firms are initially identical.

5. Extension: Consumers do not make a purchase in each period

In this Section, it is analyzed how a reduction in the frequency of purchases affects the evolution of market shares. The idea that not every consumer makes a new purchase in each period adds an important aspect of real-world markets to the model.

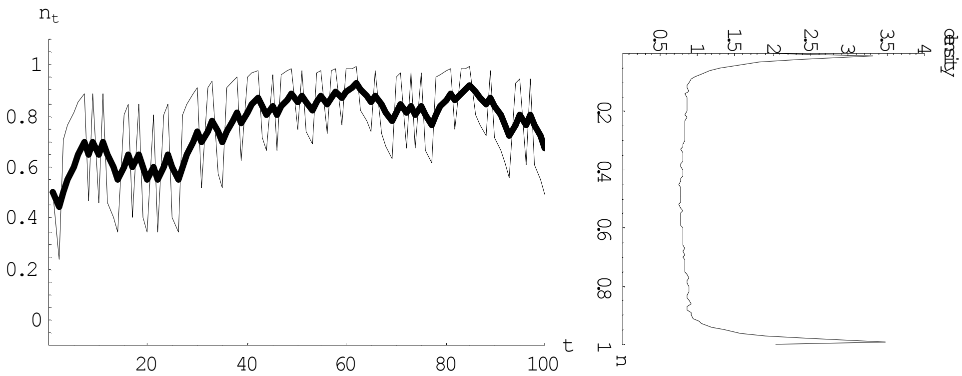
Let α be the (average) frequency of purchases of a consumer in this market. E.g., $\alpha = 1/2$ means that consumers (on average) return to the market every second period. The parameter α can also be interpreted as a measure for the degree of durability of the good sold in this market. As in Section 2, let us assume that the mass of consumers who return to the market in each period is normalized to 1.[33] Let Ω_t be the mass of consumers who purchased firm 1's product when they made their last purchase in this market (evaluated at the beginning of period t). Among these consumers, let n_t be the mass of those consumers who return to the market in period t: $n_t = \alpha \cdot \Omega_t$. Note, that n_t differs from firm 1's market share in period $t-1$ ($D_{1,t-1}$) when $\alpha < 1$. Under the above assumptions, we obtain the following law of motion: $\Omega_t = (1-\alpha)\Omega_{t-1} + D_{1,t-1}$. Using $n_t = \alpha \cdot \Omega_t$, this can be written as:

$$n_t = (1-\alpha)n_{t-1} + \alpha D_{1,t-1} \tag{23}$$

(23) replaces the simple relation $n_t = D_{1,t-1}$ assumed in Section 2. Otherwise, the model and the solution procedure remain unchanged. Note, however, that for $\alpha < 1$, the market shares are (by (23)) more volatile than the state variable n_t (a reduction in the frequency of repeat purchasing α adds inertia to the state variable).

Figure 12 shows a simulation and the invariant distribution of market shares under the stochastic benchmark, for $\lambda = 0.95$, $\rho = 0$, and $\alpha = 0.2$.[34]

Figure 12: Simulated evolution and invariant distribution of market shares, stochastic benchmark; $\lambda = 0.95$, $\rho = 0$, $\alpha = 0.2$ (thick curve: state n_t, thin: $D_{1,t}$)



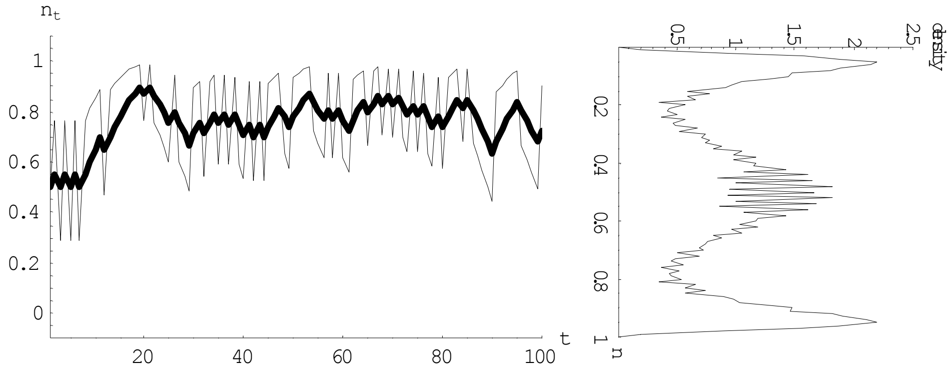
The thin curve in Figure 12 shows firm 1's realized market shares. It fluctuates around the thicker curve that shows the evolution of the state variable n_t. The comparison of Figure 12

[33] Hence, when $\alpha < 1$, the total mass of consumers (including those who do *not* make a purchase in the current period) is greater than 1.

[34] Note, that the invariant distribution of the state variable has a different shape (not shown).

and Figure 3 (same parameters except α) shows that market shares are (on average) less skewed for $\alpha = 0.2$ than for $\alpha = 1$. Overall, there is more inertia for lower values of α.[35]
Figure 13 shows a simulation of the full dynamic game for the same parameters and $\delta = 0.7$.

Figure 13: Simulated evolution and invariant distribution of market shares, full dynamic game; $\lambda = 0.95$, $\rho = 0$, $\alpha = 0.2$, $\delta = 0.7$



The evolution of market shares in Figure 13 is qualitatively similar to the one under the stochastic benchmark (Figure 12). The invariant distribution reveals an interesting result: market shares near the extremes and even splits of the market occur more frequently than market shares near 0.8 (or 0.2). This is related to the fact that the invariant distribution of the state has two peaks located near 0.2 and 0.8 (not shown). Since firm 1's realized market shares fluctuate around the state, we observe two areas with *low* probabilities around 0.2 and 0.8 in Figure 13. Intuitively, when firm 1's customer base size n_t is large, it prefers to exploit the locked-in consumers by charging a high price. Hence, it tends to lose market shares. However, when n becomes "too small" (say, below 0.8), firm 1 starts to *defend* its dominant position in the market. As a result, the state often remains near 0.8 (or 0.2) for many consecutive periods.
In a final excursion, let us briefly discuss a situation where a firm tries to enter a market characterized by a high degree of inertia. Figure 14 simulates entry dynamics for a market where most consumers either communicate via word-of-mouth, or are of the ignorant type.

[35] The comparison of the invariant distributions in Figures 3 and 12 shows that they can become more regular shaped when α is reduced (this holds for large values of λ). As a result, the iteration converges also for larger values of δ when α is sufficiently small. However, when λ is *small*, the situation is reversed. While for $\alpha = 1$, the iteration, then, converges for all values of δ, for $\alpha < 1$, the invariant distribution becomes irregular, and the iteration does not converge if δ is too large.

Figure 14: Simulated evolution of market shares, full dynamic game; $\lambda = \rho = 0.49$, $\alpha = 0.1$, $\delta = 0.5$

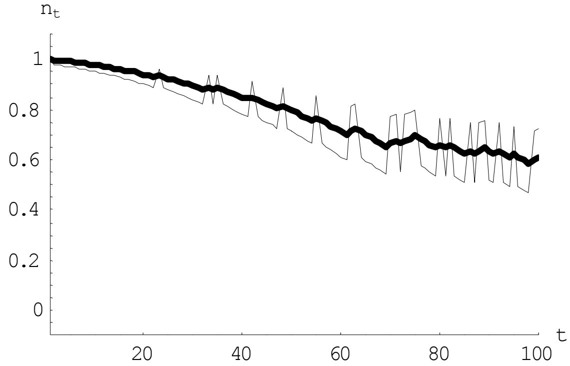


Figure 14 nicely illustrates that entry into a market with poorly informed consumers and a low frequency of purchases can be rather time consuming. The entrant only gradually captures a greater share of the market by continuously undercutting the incumbent's price. The state gradually approaches values near 1/2, where most of its probability mass is located for the given parameter values (not shown). Market shares, then, start to fluctuate around the state.

6. Conclusion

Authors of search models often restrict their attention to situations where consumers do not return to the market, and knowledge from previous consumption experiences does not exist. Hence, they use a static modeling framework, which is appropriate in the absence of an intertemporal link in demand. However, there is empirical evidence that market splits (even or skewed) can have a high persistence over time.[36] This paper introduced an intertemporal link in demand, by assuming that consumers are repeat purchasers who possess information about the supplier that they visited in the previous period. Combined with the price dispersion generated by the firms' use of mixed pricing strategies, this leads to rather non-trivial market share dynamics. The paper showed that when many consumers search actively, market shares are volatile, and firms have little incentive to invest in a customer base. Market shares, then, fluctuate between the extremes of the market share space. When many consumers are of the 'ignorant type' and remain locked-in at their previous supplier, market shares are sticky and tend to be evenly distributed most of the time. However, as the valuation of future profits increases, a larger customer base becomes valuable. Hence, a firm with a dominant position in the market may become a more aggressive player who vigorously defends its position. In this case, a tendency towards more skewed market splits emerges. This effect is especially pronounced when many consumers communicate via word-of-mouth. In this case, the

[36] See, e.g., Pakes (1987).

'popularity weighting property' of word-of-mouth communication plays an important role. When future profits are not fully discounted, a skewed split of the market can, thus, be the "natural" outcome even though firms are initially identical. In contrast, when firms maximize only current profits, a firm with a smaller customer base is generally the more aggressive player. This yields a tendency towards more even splits of the market.

In an extension of the model, it was shown that, if consumers do not return to the market in each period, there is more inertia in the market shares. This case was used to simulate entry dynamics. The dynamics illustrated that entry into a market with poorly informed consumers can be a rather time consuming process. An entrant must undercut the incumbent's price for many consecutive periods in order to obtain a sizable share of the market.

Appendix

Proof of Proposition 1:

Proof of the first claim ("there is no Markov perfect equilibrium that comprises pure strategies for any $n \in [0,1]$") by contradiction. Suppose there is an equilibrium that assigns pure strategies p_1 and p_2 to a state $n \in [0,1]$. It must be shown that, for any p_1 and p_2, at least one firm has an incentive to deviate. If $p_1 = p_2 \equiv p$ and $p > 0$, either firm would benefit from marginally undercutting the common price p. This leads to a discontinuous rise in demand that, for sufficiently low δ, more than compensates for (potential) future losses resulting from the increase in the size of the firm's customer base.[37] The case $p_1 = p_2 \equiv p$ and $p \le 0$ can not be an equilibrium because, if it were an equilibrium, the state n would remain constant, so firms choose $p_1 = p_2 \equiv p$ in *all* periods and total discounted profits are non-positive (note: a deviation e.g. to the monopoly price 1 yields a positive profit to a firm with a positive customer base size, and there is at least one such firm). If $p_1 \neq p_2$ and $p_1, p_2 < 1$, the high-priced firm would benefit from deviating to the monopoly price because current demand and, thus, the state next period are not affected. There can be no pure strategy equilibrium where $p_i = 1$ and $p_{j \neq i} < 1$, as the low-priced firm would benefit from deviating to a higher price, or the high-priced firm from marginally undercutting the lower price (or both).

Proof of the second claim ("both equilibrium price distribution functions $F(n, p)$ and $F(1-n, p)$ have the same support (for any given n), the convex set of prices p from

[37] Note: an increase in the size of a firm's customer base can lead to a *reduction* in profit, unless the value function is monotonically increasing, but this can not be imposed here.

$\underline{p}(n) < 1$ to 1"): Let S_1 and S_2 be the supports of, respectively, $F(n, p)$ and $F(1-n, p)$. Suppose, $S_1 \neq S_2$ for a given value of n. Therefore, there is a price $\tilde{p} < 1$, with $\tilde{p} \in S_i$ but $\tilde{p} \notin S_j$ ($i, j \in \{1, 2\}$, $j \neq i$). This can not be an equilibrium because, since $\overline{S}_j$ is open, there exists a price $p > \tilde{p}$ with $p \notin S_j$ that yields the same expected demand to firm i in the current period as $\tilde{p}$, but a higher profit if δ is sufficiently small. This holds if p is above the maximum of S_j (if the maximum is below 1), below the minimum of S_j, or within some intermediate range that is not part of S_j when S_j is not convex. Therefore, $S_1 = S_2 \equiv S$. The maximum of S must be the monopoly price, because otherwise, each firm would benefit from deviating to the monopoly price. This yields the same demand as the maximum of S but a higher profit. The minimum of the support, $\underline{p}$, is smaller than 1 since there is no pure strategy equilibrium. Furthermore, S is convex. Suppose to the contrary that there is an intermediate range that is not part of S. Firm i's expected demand would be constant over this range, but expected profit would be increasing. Therefore, expected profit would be higher in the upper interval of S. This can not be an equilibrium.

Proof of the third claim ("at most one firm attaches positive probability mass to any single price, and if so, the mass point is located at the monopoly price"): Suppose both firms attach positive probability mass to some identical price level p in $S = [\underline{p}, 1]$ when the current state is n. Each firm would, then, benefit from shifting its mass point to a price level marginally below p because this leads to a discontinuous rise in current expected demand, which always benefits the firm if δ is sufficiently small. Strategies containing a single mass point at some price p in the interval $(\underline{p}, 1)$ can not be an equilibrium either since the competitor's expected demand would fall discontinuously at p. There can be no equilibrium where the distribution function of one firm contains a mass point at $p = \underline{p} > 0$ as the competitor's current expected profit would be larger at prices marginally below $\underline{p}$ than at prices above $\underline{p}$. This can not be an equilibrium since prices below $\underline{p}$ are not part of the support. There can be no equilibrium where the distribution function of one firm contains a mass point at $p = \underline{p} \leq 0$ as the firm would benefit from shifting the mass point to some higher price level. *Q.E.D.*

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