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Unawareness, Beliefs and Games*

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Abstract

We define a generalized state-space model with interactive unawareness and probabilistic beliefs. Such models are desirable for many potential applications of asymmetric unawareness. We develop Bayesian games with unawareness, define equilibrium, and prove existence. We show how equilibria are extended naturally from lower to higher awareness levels and restricted from higher to lower awareness levels. We use our unawareness belief structure to show that the common prior assumption is too weak to rule out speculative trade in all states. Yet, we prove a generalized "No-trade" theorem according to which there can not be common certainty of strict preference to trade. Moreover, we show a generalization of the "No-agreeing-to-disagree" theorem.

Keywords: unawareness, awareness, type-space, Bayesian games, incomplete information, equilibrium, common prior, agreement, speculative trade, interactive epistemology.

JEL-Classifications: C70, C72, D80, D82.

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1 Introduction

Unawareness is probably the most common and most important kind of ignorance. Business people invest most of their time not in updating prior beliefs, and crossing out states of the world that they previously assumed to be possible. Rather, their efforts are mostly aimed at exploring unmapped terrain, trying to figure out business opportunities that they could not even have spelled out before. More broadly, every book we read, every new acquaintance we make, expands our horizon and our language, by fusing it with the horizons of those we encounter, turning the world more intelligible and more meaningful to us than it was before (Gadamer, 1960).

With this in mind, we should not be surprised that the standard state spaces aimed at modeling knowledge or certainty are not adequate for capturing unawareness (Dekel, Lipman and Rustichini, 2001). Indeed, more elaborate models are needed (Fagin and Halpern, 1988, Modica and Rustichini, 1994, 1999, Halpern, 2001). In all of these models, the horizon of propositions the individual has in her disposition to talk about the world is always a genuine part of the description of the state of affairs.

Things become even more intricate when several players are involved. Different players may not only have different languages. On top of this, each player may also form a belief on the extent to which other players are aware of the issues that she herself has in mind. And the complexity continues further, because the player may be uncertain as to the sub-language that each other player attributes to her or to others; and so on.

Heifetz, Meier and Schipper (2006a) showed how an unawareness structure consisting of a *lattice of spaces* is adequate for modeling mutual unawareness. Every space in the lattice captures one particular horizon of meanings or propositions. Higher spaces capture wider horizons, in which states correspond to situations described by a richer vocabulary. The join of several spaces – the lowest space at least as high as every one of them – corresponds to the fusion of the horizons of meanings expressible in these spaces.

In a companion work (Heifetz, Meier and Schipper, 2006b), we showed the precise sense in which such unawareness structures are adequate and general enough for modeling mutual unawareness. We put forward an axiom system, which extends to the multi-player case a variant of the axiom system of Modica and Rustichini (1999). We then showed how the collections of all maximally-consistent sets of formulas in our system form a canonical unawareness structure.[1] In a parallel work, Halpern and Rêgo (2005) devised another sound and complete axiomatization for our class of unawareness structures.

In this paper we extend unawareness structures so as to encompass probabilistic beliefs (Section 2) rather than only knowledge or ignorance. The definition of types (Definition 1), and the way beliefs relate across different spaces of the lattice, is a non-trivial modification of the coherence conditions for knowledge operators in unawareness structures, as formulated in Heifetz, Meier and Schipper (2006a).

With unawareness type spaces in hand, we can define Bayesian games. Here again, the definition of a strategy is not obvious. Consider a type τ with a narrow horizon, and two other

[1]Each space in the lattice of this canonical unawareness structure consists of the maximally consistent sets of formulas in a sub-language generated by a subset of the atomic propositions.

types τ', τ'' with a wider horizon, that agree with the quantitative beliefs of τ regarding the aspects of reality of which τ is aware; the beliefs of τ' and τ'' differ only concerning dimensions of reality that τ does not conceive. Should the action taken by τ necessarily be some average of the actions taken by τ' and τ''? We believe that conceptually, the answer to this question is negative. When the player conceives of more parameters (e.g. motives for saving) as relevant to her decision, her optimal action (e.g. "invest in bonds" or "invest in stocks") need not be related to her optimal decision (e.g. "go shopping") when that parameter is not part of the vocabulary with which she conceives the world.[2]

The next step is to define Bayesian equilibrium. With finitely or countably many states, existence follows from standard arguments.[3] Unawareness, however, introduces a new aspect to the construction of equilibrium, namely "the tyranny of the unaware": A type who concieves of only few dimensions of reality does not have in mind types of other players with a wider horizon, so the optimal action of this type *does not depend* on the actions of these wider-horizon types. Those types, however, who assign a positive probability to this narrow-minded type, must take its action into account when optimizing.

In Section 4 we define the notion of a common prior. Conceptually, a prior of a player is a convex combination of (the beliefs of) her types (see e.g. Samet, 1998). If the priors of the different players coincide, we have a common prior. A prior of a player induces a prior on each particular space in the lattice, and if the prior is common to the players, the induced prior on each particular space is common as well.

What are the implications of the existence of a common prior? First, we extend an example from Heifetz, Meier and Schipper (2006a) and show that *speculative trade* is compatible with the existence of a common prior. This need not be surprising if one views unawareness as a particular kind of delusion, since we know that with deluded beliefs, speculative trade is possible even with a common prior (Geanakoplos, 1989). Nevertheless, we show that under a mild non-degeneracy condition, a common prior is *not compatible* with common certainty of *strict preference* to carry out speculative trade. That is, even though types with limited awareness are, in a particular sense, deluded, a common prior precludes the possibility of common certainty of the event that based on private information players are willing to engage in a zero-sum bet with *strictly positive* subjective gains to everybody. This is so because unaware types are "deluded" only concerning aspects of the world outside their vocabulary, while a common prior captures a prior agreement on the likelihood of whatever the players do have a common vocabulary. An implication of this generalized no-trade theorem is that arbitrary small transaction fees rule out speculative trade under unawareness. We complement this result be generalizing Aumann's (1976) "No-Agreeing-to-disagree" result to unawareness belief structures.

There is a growing literature on unawareness both in economics and computer science. The independent parallel work of Sadzik (2006) is closest to ours. Building on our earlier work, Heifetz, Meier and Schipper (2006a), he presents a framework of unawareness with probabilistic beliefs in which the common prior on the upmost space is a primitive. In contrast, we take types as primitives and define a prior on the entire unawareness belief structure as a convex

[2]This is a crucial point in which our definition of a strategy diverges from the one in the parallel work of Sadzik (2006). This paper confines attention to a setting with a common prior, that we discuss as a special case.

[3]Recall that in standard type spaces (with no unawareness), a Bayesian equilibrium need not exist even in "non-pathological" spaces with a continuum of states (Simon 2003).

combination of the type's beliefs. Sadzik (2006) also considers Bayesian games with unawareness, but his definition of Bayesian strategy and consequently the notion of equilibrium differs from ours. As argued above, we do not confine actions of a type with a narrow horizon to be some average of actions of the corresponding types with a wider horizon, a restriction made in Sadzik (2006). As a result, in our notion of Bayesian equilibrium every type maximizes and is certain that every other type that she is aware of maximizes as well, while in the equilibrium of actions proposed in Sadzik (2006) a type may believe that another player is irrational. Sadzik (2006) does not allow for unawareness of players, while we do (see the appendix).

A purely syntactic framework with unawareness is presented by Feinberg (2004, 2005). He applies it to games with unawareness of actions but complete information. In the appendix, we discuss an interesting example due to Feinberg (2005) and demonstrate that higher order awareness of unawareness in Feinberg (2005) corresponds to higher order belief of unawareness in our model. In a framework similar to Feinberg (2004, 2005), Čopič and Galeotti (2006) study two-player games with either unawareness of actions or unawareness of types (with a prior as a primitive). Yet, they postulate that in equilibrium beliefs over actions and payoffs must correspond to the true joint distribution over own payoffs and the opponent's actions.

Both Halpern and Rêgo (2006) and Li (2006b) present models of extensive form games with unawareness and analyze solution concepts for them.[4] Modica (2000) studies the updating of probabilities and argues that new information may change posteriors more if it implies also a higher level of awareness. A dynamic framework for a single decision maker with unawareness is introduced by Grant and Quiggin (2006). Ewerhart (2001) studies the possibility of agreement under a notion of unawareness different from the aforementioned literature. Lastly, Ahn and Ergin (2006) consider explicitly more or less fine descriptions of acts and characterize axiomatically a partition-dependent subjective expected utility representation. Since the set of all partitions of a state-space forms a complete lattice, their approach suggests a decision theoretic foundation of subjective probabilities on our lattice structure.

In the following section we present our interactive unawareness belief structure. In Section 3 Bayesian games with unawareness are developed. In Section 4 we define a common prior and investigate agreement and speculation under unawareness. Some further properties of our unawareness belief structure are relegated to the appendix, which also contains a generalization of Bayesian games in order to include unawareness of actions and players. Proofs are relegated to the appendix as well. In a separate appendix, Meier and Schipper (2007), we extend the "No-trade" theorem to infinite unawareness structures.

2 Model

2.1 State-Spaces

Let $\mathcal{S} = \{S_\alpha\}_{\alpha \in \mathcal{A}}$ be a complete lattice of disjoint *state-spaces*, with the partial order $\preceq$ on $\mathcal{S}$. If S_α and S_β are such that $S_\alpha \succeq S_\beta$ we say that "S_α is more expressive than S_β – states of S_α

[4] Li (2006b) is based on her earlier work, Li (2006a).

describe situations with a richer vocabulary than states of S_β ".[5] Denote by $\Omega = \bigcup_{\alpha \in \mathcal{A}} S_\alpha$ the union of these spaces. Each $S \in \mathcal{S}$ is a measurable space, with a σ-field $\mathcal{F}_S$.

Spaces in the lattice can be more or less "rich" in terms of facts that may or may not obtain in them. The partial order relates to the "richness" of spaces. The upmost space of the lattice can be interpreted as the "objective" state-space. Its states encompass full descriptions from the point of view of the modeler.

2.2 Projections

For every S and S' such that $S' \succeq S$, there is a measurable surjective projection $r_S^{S'} : S' \to S$, where r_S^S is the identity. ("$r_S^{S'}(\omega)$ is the restriction of the description ω to the more limited vocabulary of S.") Note that the cardinality of S is smaller than or equal to the cardinality of S'. We require the projections to commute: If $S'' \succeq S' \succeq S$ then $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$. If $\omega \in S'$, denote $\omega_S = r_S^{S'}(\omega)$. If $D \subseteq S'$, denote $D_S = \{\omega_S : \omega \in D\}$.

Projections "translate" states in "more expressive" spaces to states in "less expressive" spaces by "erasing" facts that can not be expressed in a lower space.

2.3 Events

Denote $g(S) = \{S' : S' \succeq S\}$. For $D \subseteq S$, denote $D^\uparrow = \bigcup_{S' \in g(S)} \left(r_S^{S'}\right)^{-1}(D)$. ("All the extensions of descriptions in D to at least as expressive vocabularies.")

An *event* is a pair (E, S), where $E = D^\uparrow$ with $D \subseteq S$, where $S \in \mathcal{S}$. D is called the *base* and S the *base-space* of (E, S), denoted by $S(E)$. If $E \neq \emptyset$, then S is uniquely determined by E and, abusing notation, we write E for (E, S). Otherwise, we write $\emptyset^S$ for $(\emptyset, S)$. Note that not every subset of Ω is an event.

Some fact may obtain in a subset of a space. Then this fact should be also "expressible" in "more expressive" spaces. Therefore the event contains not only the particular subset but also its inverse images in "more expressive" spaces.

Let Σ be the set of measurable events of Ω, i.e., $D^\uparrow$ such that $D \in \mathcal{F}_S$, for some state space $S \in \mathcal{S}$.

2.4 Negation

If $(D^\uparrow, S)$ is an event where $D \subseteq S$, the negation $\neg(D^\uparrow, S)$ of $(D^\uparrow, S)$ is defined by $\neg(D^\uparrow, S) := ((S \backslash D)^\uparrow, S)$. Note, that by this definition, the negation of a (measurable) event is a (measurable) event. Abusing notation, we write $\neg D^\uparrow := (S \setminus D)^\uparrow$. Note that by our notational convention, we have $\neg S^\uparrow = \emptyset^S$ and $\neg \emptyset^S = S^\uparrow$, for each space $S \in \mathcal{S}$. The event $\emptyset^S$ should be interpreted as a "logical contradiction phrased with the expressive power available in S." $\neg D^\uparrow$ is typically a proper subset of the complement $\Omega \setminus D^\uparrow$. That is, $(S \setminus D)^\uparrow \subsetneqq \Omega \setminus D^\uparrow$.

Intuitively, there may be states in which the description of an event $D^\uparrow$ is both expressible

[5] Here and in what follows, phrases within quotation marks hint at intended interpretations, but we emphasize that these interpretations are not part of the definition of the set-theoretic structure.

and valid – these are the states in $D^{\uparrow}$; there may be states in which this description is expressible but invalid – these are the states in $\neg D^{\uparrow}$; and there may be states in which neither this description nor its negation are expressible – these are the states in

$$\Omega \setminus \left(D^{\uparrow} \cup \neg D^{\uparrow}\right) = \Omega \setminus S\left(D^{\uparrow}\right)^{\uparrow}.$$

Thus our structure is not a standard state-space model in the sense of Dekel, Lipman, and Rustichini (1998).

2.5 Conjunction and Disjunction

If $\left\{\left(D_{\lambda}^{\uparrow}, S_{\lambda}\right)\right\}_{\lambda \in L}$ is an at most countable collection of events (with $D_{\lambda} \subseteq S_{\lambda}$, for $\lambda \in L$), their conjunction $\bigwedge_{\lambda \in L}\left(D_{\lambda}^{\uparrow}, S_{\lambda}\right)$ is defined by $\bigwedge_{\lambda \in L}\left(D_{\lambda}^{\uparrow}, S_{\lambda}\right) := \left(\left(\bigcap_{\lambda \in L} D_{\lambda}^{\uparrow}\right), \sup_{\lambda \in L} S_{\lambda}\right)$. Note, that since $\mathcal{S}$ is a *complete* lattice, $\sup_{\lambda \in L} S_{\lambda}$ exists. If $S = \sup_{\lambda \in L} S_{\lambda}$, then we have $\left(\bigcap_{\lambda \in L} D_{\lambda}^{\uparrow}\right) = \left(\bigcap_{\lambda \in L}\left(\left(r_{S_{\lambda}}^{S}\right)^{-1}\left(D_{\lambda}\right)\right)\right)^{\uparrow}$. Again, abusing notation, we write $\bigwedge_{\lambda \in L} D_{\lambda}^{\uparrow} := \bigcap_{\lambda \in L} D_{\lambda}^{\uparrow}$ (we will therefore use the conjunction symbol $\wedge$ and the intersection symbol $\cap$ interchangeably).

We define the relation $\subseteq$ between events (E, S) and (F, S'), by $(E, S) \subseteq (F, S')$ if and only if $E \subseteq F$ as sets *and* $S' \preceq S$. If $E \neq \emptyset$, we have that $(E, S) \subseteq (F, S')$ if and only if $E \subseteq F$ as sets. Note however that for $E = \emptyset^{S}$ we have $(E, S) \subseteq (F, S')$ if and only if $S' \preceq S$. Hence we can write $E \subseteq F$ instead of $(E, S) \subseteq (F, S')$ as long as we keep in mind that in the case of $E = \emptyset^{S}$ we have $\emptyset^{S} \subseteq F$ if and only if $S \succeq S(F)$. It follows from these definitions that for events E and F, $E \subseteq F$ is equivalent to $\neg F \subseteq \neg E$ only when E and F have the same base, i.e., $S(E) = S(F)$.

The disjunction of $\left\{D_{\lambda}^{\uparrow}\right\}_{\lambda \in L}$ is defined by the de Morgan law $\bigvee_{\lambda \in L} D_{\lambda}^{\uparrow} = \neg\left(\bigwedge_{\lambda \in L} \neg\left(D_{\lambda}^{\uparrow}\right)\right)$. Typically $\bigvee_{\lambda \in L} D_{\lambda}^{\uparrow} \subsetneqq \bigcup_{\lambda \in L} D_{\lambda}^{\uparrow}$, and if all D_{λ} are nonempty we have that $\bigvee_{\lambda \in L} D_{\lambda}^{\uparrow} = \bigcup_{\lambda \in L} D_{\lambda}^{\uparrow}$ holds if and only if all the $D_{\lambda}^{\uparrow}$ have the same base-space. Note, that by these definitions, the conjunction and disjunction of (at most countably many measurable) events is a (measurable) event.

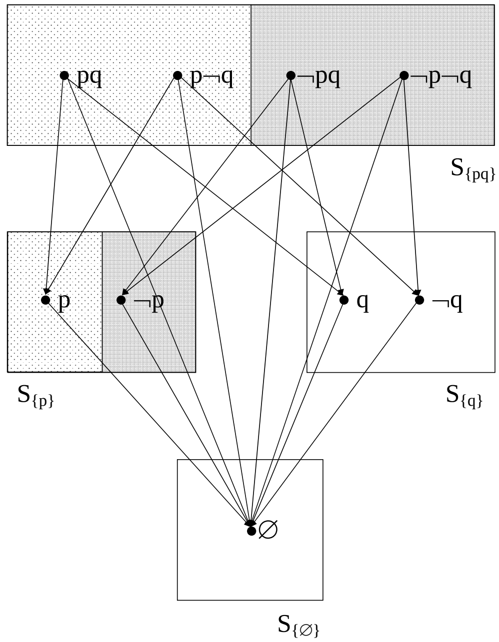
Apart from the measurability conditions, the event-structure outlined so far is analogous to Heifetz, Meier and Schipper (2006a, 2006b). An example is shown in Figure 1. It depicts a lattice with four spaces and projections. The event that p obtains is indicated by the dotted areas, whereas the grey areas illustrate the event that not p obtains.

2.6 Probability Measures

Here and in what follows, we mean by events always measurable events in Σ unless otherwise stated.

Let $\Delta(S)$ be the set of probability measures on $(S, \mathcal{F}_{S})$. We consider this set itself as a measurable space endowed with the σ-field $\mathcal{F}_{\Delta(S)}$ generated by the sets $\{\mu \in \Delta(S) : \mu(D) \geq p\}$, where $D \in \mathcal{F}_{S}$ and $p \in [0, 1]$.

Figure 1: Event Structure



2.7 Marginals

For a probability measure $\mu \in \Delta(S')$, the marginal $\mu_{|S}$ of μ on $S \preceq S'$ is defined by

$$\mu_{|S}(D) := \mu\left(\left(r_S^{S'}\right)^{-1}(D)\right), \quad D \in \mathcal{F}_S.$$

Let S_μ be the space on which μ is a probability measure. Whenever $S_\mu \succeq S(E)$ then we abuse notation slightly and write

$$\mu(E) = \mu(E \cap S_\mu).$$

If $S(E) \npreceq S_\mu$, then we say that $\mu(E)$ is undefined.

2.8 Types

I is the nonempty set of individuals. For every individual, each state gives rise to a probabilistic belief over states in some space.

Definition 1 *For each individual $i \in I$ there is a type mapping $t_i : \Omega \to \bigcup_{\alpha \in \mathcal{A}} \Delta(S_\alpha)$, which is measurable in the sense that for every $S \in \mathcal{S}$ and $Q \in \mathcal{F}_{\Delta(S)}$ we have $t_i^{-1}(Q) \cap S \in \mathcal{F}_S$, for all $S \in \mathcal{S}$.*

We require the type mapping t_i to satisfy the following properties:

(0) Confinement: If $\omega \in S'$ then $t_i(\omega) \in \triangle(S)$ for some $S \preceq S'$.

(1) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \triangle(S)$ then $t_i(\omega_{S'}) = t_i(\omega)$.

(2) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \triangle(S')$ then $t_i(\omega_S) = t_i(\omega)_{|S}$.

(3) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega_{S'}) \in \triangle(S)$ then $S_{t_i(\omega)} \succeq S$.

$t_i(\omega)$ represents individual i's belief at state ω. Properties (0) to (3) guarantee the coherence of belief and awareness down the lattice structure. *Confinement* means that at any given state $\omega \in \Omega$ an individual's belief is concentrated on states that "are all described with the same vocabulary - the vocabulary available to the individual at ω. This vocabulary may be less expressive than the vocabulary used to describe statements in the state ω."

Properties (1) to (3) compare the types of an individual in a state ω and its projection to ω_S. Property (1) and (2) means that at the projected state ω_S the individual believes everything she believes at ω given that she is aware of it at ω_S. Property (3) means that at ω an individual can not be unaware of an event that she is aware of at the projected state ω_S.

Define[6]

$$Ben_i(\omega) := \left\{\omega' \in \Omega : t_i(\omega')_{|S_{t_i(\omega)}} = t_i(\omega)\right\}.$$

This is the set of states at which individual i's type or the marginal thereof coincides with her type at ω. Such sets are events in our structure:

Remark 1 *For any $\omega \in \Omega$, $Ben_i(\omega)$ is an $S_{t_i(\omega)}$-based event.*

Note that $Ben_i(\omega)$ may not be measurable.

Assumption 1 *If $Ben_i(\omega) \subseteq E$, for an event E, then $t_i(\omega)(E) = 1$.*

This assumption implies introspection (Property (v)) in Proposition 7. Note, that if $Ben_i(\omega)$ is measurable, then Assumption 1 implies $t_i(\omega)(Ben_i(\omega)) = 1$.

Definition 2 *We denote by $\underline{\Omega} := \left\langle \mathcal{S}, \left(r_{S_\beta}^{S_\alpha}\right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I} \right\rangle$ an interactive unawareness belief structure.*

2.9 Awareness and Unawareness

The definition of awareness is analogous to the definition in unawareness knowledge structures (see Remark 6 in Heifetz, Meier and Schipper, 2006b).

Definition 3 *For $i \in I$ and an event E, define the awareness operator*

$$A_i(E) := \{\omega \in \Omega : t_i(\omega) \in \Delta(S), S \succeq S(E)\}$$

if there is a state ω such that $t_i(\omega) \in \Delta(S)$ with $S \succeq S(E)$, and by

$$A_i(E) := \emptyset^{S(E)}$$

otherwise.

[6] The name "Ben" is chosen analogously to the "ken" in knowledge structures.

An individual is aware of an event if and only if his type is concentrated on a space in which the event is "expressible."

Proposition 1 *If E is an event then $A_i(E)$ is an $S(E)$-based event.*

This proposition shows that the set of states in which an individual is aware of an event is indeed an event in our structure. Moreover, the operator is convenient to work with since the event $A_i(E)$ has the same base-space as the event E.

Unawareness is naturally defined as the negation of awareness:

Definition 4 *For $i \in I$ and an event E, the unawareness operator is now defined by*

$$U_i(E) = \neg A_i(E).$$

Note that the definition of our negation and Proposition 1 imply that if E is an event, then $U_i(E)$ is an $S(E)$-based event.

Note further that Definition 3 and 4 apply also to events that are not necessarily measurable.

An example of unawareness is presented in Example 1, Figure 2. For instance, at state ω_2, Nikolai (intermitted ellipses) is unaware of the event S since his type is concentrated on S', where $S' \preceq S$. Yet, he is aware of the event $S'^{\uparrow}$.

2.10 Belief

The p-belief-operator is defined as usual (see for instance Monderer and Samet, 1989):

Definition 5 *For $i \in I$, $p \in [0,1]$ and an event E, the p-belief operator is defined, as usual, by*

$$B_i^p(E) := \{\omega \in \Omega : t_i(\omega)(E) \geq p\},$$

if there is a state ω such that $t_i(\omega)(E) \geq p$, and by

$$B_i^p(E) := \emptyset^{S(E)}$$

otherwise.

Proposition 2 *If E is an event then $B_i^p(E)$ is an $S(E)$-based event.*

This proposition shows that the set of states in which an individual believes an event with probability at least p is an event in our structure that has the same base-space as the event E.

The p-belief operator has the standard properties stated in **Proposition 7** in the appendix.

Dekel, Lipman and Rustichini (1998) showed that in a standard state-space unawareness must be trivial, even if the belief operator satisfies only very weak properties. In contrast, we show in **Proposition 8** (see appendix) that the strong notion of p-belief (in particular also probability one belief) allows for all properties of unawareness that have been proposed in the literature.

In **Proposition 10** in the appendix, we state some multi-person properties of awareness and belief. For instance, we show that if an individual is aware of an event E, then she can also conceive of that others are aware of the event E. Moreover, we show that common awareness and mutual awareness coincide. That is, if everybody is aware of an event, then everybody can conceive of that everybody is aware of the event, everybody is aware of that, etc.

3 Bayesian Games with Unawareness

In this section, we generalize strategic games with incomplete information a là Harsanyi (1967/68) and Mertens and Zamir (1985, Section 5) to include also unawareness. For simplicity, we consider first Bayesian games with unawareness in which every player is aware of all of her and other's actions, and all the players. In the appendix we generalize our theory to allow also for unawareness of actions and players. For notational convenience, we restrict ourselves in this section to a finite set of players, finite sets of actions, finite state-spaces, and assume that for each $S \in \mathcal{S}$, $\mathcal{F}_S = 2^S$.

Definition 6 *A Bayesian game with unawareness of events consists of an unawareness belief structure* $\underline{\Omega} = \left\langle \mathcal{S}, \left(r^{S_\alpha}_{S_\beta}\right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I} \right\rangle$ *that is augmented by a tuple* $\left\langle (M_i)_{i \in I}, (u_i)_{i \in I} \right\rangle$ *defined as follows: For each player* $i \in I$*, there is*

(i) a finite nonempty set of actions M_i*, and*

(ii) a utility function $u_i : \left(\prod_{i \in I} M_i\right) \times \Omega \longrightarrow \mathbb{R}$.

The interpretation is as follows: At the beginning of a game, a state $\omega \in \Omega$ is realized. Player i does not observe the state but receives a signal $t_i(\omega)$ that provides her with some information about the state or projections thereof to lower spaces. I.e., if ω obtains, player i is of type $t_i(\omega)$. This signal is a belief about the likelihood of events on a certain space. A player's utility depends on her action, the actions chosen by other players as well as the state. Since players may be uncertain about the state ω, we assume below that the player's preference is represented by the expected value of the utility function on action-profiles of players and states, where the expectation is taken with respect to the player i's type $t_i(\omega)$ and the types' mixed strategies. This game allows for unawareness of possibly payoff relevant events.

Let $\Delta(M_i)$ be the set of mixed strategies for player $i \in I$, that is, the set of probability distributions on the finite set M_i.

Definition 7 *A strategy of player* i *in a Bayesian game with unawareness of events is a function* $\sigma_i : \Omega \longrightarrow \Delta(M_i)$ *such that for all* $\omega \in \Omega$,

(i) $\sigma_i(\omega) \in \Delta(M_i)$,

(ii) $t_i(\omega) = t_i(\omega')$ *implies* $\sigma_i(\omega) = \sigma_i(\omega')$.

A strategy specifies for each player and state a probability distribution over her set of her actions. In standard Bayesian games without unawareness, one interpretation of a strategy assumes an ex-ante point of view of the player before she knows her type. This interpretation is misleading in a game with unawareness, since if a player is aware of all her types ex-ante she should be also aware of all types interim, i.e., after learning her type (and her awareness). Hence, in the case of unawareness, the ex-ante notion of strategy is a useful construct for the game theorist rather than for a player.

In Bayesian games with unawareness we subscribe to a second interpretation of Bayesian strategy from an interim point of view: Given a player i and type $t_i(\omega)$, she has an "awareness level" $S_{t_i(\omega)} \in \mathcal{S}$. That is, she can consider strategies of her opponents in $l(S_{t_i(\omega)})$, where $l(S) := \{S' \in \mathcal{S} : S' \preceq S\}$ is the complete sublattice of $\mathcal{S}$ with S being the upmost space. This interpretation is sound precisely because of Propositions 4 and 5 below: To best-respond to the strategies of the other player-types, a type of a player needs only to reason about the strategies of player-types that she is aware of. Only strategies of these player-types enter in her utility maximization problem.

Denote $\sigma_{S_{t_i(\omega)}} := \left((\sigma_j(\omega'))_{j \in I} \right)_{\omega' \in S_{t_i(\omega)}}$. A component $\sigma_j(\omega')$ of the strategy profile $\sigma_{S_{t_i(\omega)}}$ is the strategy of the *player-type* $(j, t_j(\omega'))$. $\sigma_{S_{t_i(\omega)}}$ is the profile of all player-types' strategies.

The *expected utility* of player-type $(i, t_i(\omega))$ from the strategy profile $\sigma_{S_{t_i(\omega)}}$ is given by

$$U_{(i,t_i(\omega))}(\sigma_{S_{t_i(\omega)}}) := \int_{\omega' \in S_{t_i(\omega)}} \sum_{m \in \prod_{j \in I} M_j} \prod_{j \in I} \sigma_j(\omega')\,(m_j) \cdot u_i\left((m_j)_{j \in I}, \omega' \right) dt_i(\omega)(\omega'). \tag{1}$$

$\sigma_j(\omega')(m_j)$ is the probability with which the player-type $(j, t_j(\omega'))$ plays the action $m_j \in M_j$. $\prod_{j \in I} \sigma_j(\omega')\,(m_j)$ is the joint probability with which the action profile $m = (m_j)_{j \in I}$ is played by the players. This action profile gives the utility $u_i\left((m_j)_{j \in I}, \omega'\right)$ to player i in state ω'. The term $\sum_{m \in \prod_{j \in I} M_j} \prod_{j \in I} \sigma_j(\omega')\,(m_j) \cdot u_i\left((m_j)_{j \in I}, \omega' \right)$ is player i's expected utility from the strategy profile $(\sigma_j(\omega'))_{j \in I}$ at the state ω'. However, at a state ω, the player, in general, does not know the state, but only his type $t_i(\omega)$, and so he evaluates his utility with the expectation with respect to the probability measure $t_i(\omega)$.

Definition 8 (Equilibrium) *An* equilibrium *of a Bayesian game with unawareness of events* $\left\langle \mathcal{S}, \left(r^{S_\alpha}_{S_\beta} \right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I}, (M_i), (u_i) \right\rangle$ *is a Nash equilibrium of the strategic game defined by:*

(i) $\{(i, t_i(\omega)) : \omega \in \Omega$ *and* $i \in I\}$ *is the set of players,*

and for each player $(i, t_i(\omega))$,

(ii) the set of mixed strategies is $\Delta(M_i)$, *and*

(iii) the utility function is given by equation (1).

An equilibrium of a Bayesian game with unawareness is a Nash equilibrium of a strategic game in which types of players are the "players". The actions available to the type of player i at state ω are the actions of player i. The utility function of the type of player i at ω is the expected utility function, given player i's awareness and belief over states at ω. In an equilibrium of a

Bayesian game with unawareness of events, the type of every player chooses an optimal mixture of actions, given her awareness, belief and the choices of the types of the other players she is aware of. This is analogous to equilibrium in Bayesian games without unawareness.

Proposition 3 (Existence) *Let* $\left\langle \mathcal{S}, \left(r^{S_\alpha}_{S_\beta}\right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I}, (M_i), (u_i) \right\rangle$ *be a Bayesian game with unawareness of events. If* I*,* Ω*, and* $(M_i)_{i \in I}$ *are finite, then there exists an equilibrium.*

Proof. By Nash's (1950) theorem. □

Note that contrary to an ordinary Bayesian game, the game is not "common knowledge" among the players. Let $\left\langle \mathcal{S}, \left(r^{S_\alpha}_{S_\beta}\right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I}, (M_i), (u_i) \right\rangle$ be a Bayesian game with unawareness of events. At $\omega \in \Omega$, the game conceived by player i is $\left\langle l(S_{t_i(\omega)}), \left(r^{S_\alpha}_{S_\beta}\right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I}, (M_i), (u_i) \right\rangle$, where the lattice of spaces is replaced with the sublattice $l(S_{t_i(\omega)})$ with $S_{t_i(\omega)}$ as the upmost space, $\left(r^{S_\alpha}_{S_\beta}\right)$ are restricted to $S_\alpha, S_\beta \in l(S_{t_i(\omega)})$, and accordingly, the domains of the t_i and u_i are restricted to $\bigcup_{S \in l(S_{t_i(\omega)})} S$. Type $t_i(\omega)$ of player i can conceive of all events expressible in the spaces of the sublattice $l(S_{t_i(\omega)})$. For $S \in \mathcal{S}$, we call $\left\langle l(S), \left(r^{S_\alpha}_{S_\beta}\right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I}, (M_i), (u_i) \right\rangle$ the *S-partial Bayesian game with unawareness of events.*

The following proposition shows that we can naturally extend equilibria from "lower awareness levels to higher awareness levels" by taking the equilibrium strategies at the "lower awareness levels" fixed and looking for a fixed point at "higher awareness levels".

Proposition 4 ("Upwards Induction") *Given a Bayesian game with unawareness of events* $\left\langle \mathcal{S}, \left(r^{S_\alpha}_{S_\beta}\right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I}, (M_i), (u_i) \right\rangle$*, define for* $S', S'' \in \mathcal{S}$ *with* $S' \preceq S''$ *the* S'*-partial (resp.* S''*-partial) Bayesian game with unawareness of events. If* I*,* Ω*, and* $(M_i)_{i \in I}$ *are finite, then for every equilibrium of the* S'*-partial Bayesian game, there is an equilibrium of the* S''*-partial Bayesian game in which equilibrium strategies of player-types in* $\{(i, t_i(\omega)) : \omega \in \bigcup_{S \in l(S')} S \text{ and } i \in I\}$ *are identical with the equilibrium strategies in the* S'*-partial Bayesian game.*

This proposition suggests a procedure for constructing equilibria in Bayesian games with unawareness. We start with an equilibrium in the $\hat{S}$-partial Bayesian game with unawareness, where $\hat{S}$ denotes the greatest lower bound space (the meet) of the lattice, and extend it step-by-step to higher spaces by finding a fixed-point taking the strategies of player-types in the respective lower spaces as given.

For some strategic situations, Proposition 4 suggests that players which are unaware may have commitment power compared to players with a "higher awareness level". This is so because types with "lower awareness levels" do not react to types of which they are unaware. Types with "higher awareness" must take strategies of types with "lower awareness" as given. In some strategic situations, the value of awareness may be negative.

Proposition 4 may also be interpreted as what happens if players learn, i.e. become aware of some event. We can consider a player at a certain state ω, and compare her strategy with the strategy of the same player but at a state in ω's inverse image in a higher space.

Proposition 5 *Let* $\left\langle \mathcal{S}, \left(r^{S_\alpha}_{S_\beta}\right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I}, (M_i), (u_i) \right\rangle$ *be a Bayesian game with unawareness of events. Define for* $S', S'' \in \mathcal{S}$ *with* $S' \preceq S''$ *the* S'*-partial (resp.* S''*-partial) Bayesian game with unawareness of events. Then for every equilibrium of the* S''*-partial Bayesian game there is an equilibrium of the* S'*-partial Bayesian game in which the equilibrium strategies of player-types in* $\{(i, t_i(\omega)) : \omega \in \bigcup_{S \in l(S')} S$ *and* $i \in I\}$ *are identical with the equilibrium strategies of the* S''*-partial Bayesian game.*

This proposition says that we can naturally restrict an equilibrium from higher awareness levels to lower awareness. This is so, because if player-types play an equilibrium in a game that allows for "higher awareness levels", then those player-types still play optimally at "lower awareness levels given that they exist there".

We conclude this section with a simple example that touches a prime theme of unawareness: novelties, inventions and innovations.

Example 1 (The Mathematician's Dilemma) Two brilliant mathematicians, Emmy and Nicolai, consider to compete on solving a problem in mathematics. Solving the problem now rather than later is costly because there are also other unsolved problems they could try to solve. We assume that the costs of solving it now rather than later are 100K dollars for either player. Moreover, we assume that a solution to this problem is prized at 180K dollars. This is to be shared if both solve it at the same time. If only one solves it now, and the other later, then the latter gets nothing and the former 180K.

When solving the problem, any of the two mathematicians could be quite unexpectedly aware of a brilliant idea that would not only solve their problem but also prove the Riemann Hypothesis. This chance-discovery is not foreseen by anybody in the profession. Luckily the Clay Mathematics Institute of Cambridge, M.A., offers a reward of 1 million dollars for the proof of the Riemann Hypothesis. We assume that this prize is shared if both provide a proof at the same time. If one is first, then he gets the entire prize.

To model their awareness and beliefs, we consider two state-spaces S and S'. We assume that S is richer than S' in the sense that whenever a player believes some state in S, then (s)he is aware of the brilliant idea. A player's belief at each state is given by a probability distribution on one of those spaces. To be precise, consider the information structure in Figure 2. There are three states, ω_1 and ω_2 in S and ω_3 in S'. The solid (resp. dashed) lines/ovals belong to Emmy (resp. Nicolai). For instance, the solid line starting in ω_1 indicates that at state ω_1 Emmy believes with probability one that ω_2 obtained. Since $\omega_2 \in S$, she is aware of the brilliant idea. Yet, at ω_2, Nicolai believes with probability one that $\omega_3 \in S'$ obtained, which means that at ω_2, Nicolai is unaware of the brilliant idea. It also means that at ω_2, and hence at ω_1, Emmy believes that Nicolai is unaware of the brilliant idea. Note however, that at ω_1 Nicolai is aware of the brilliant idea, he believes that Emmy is aware of it, and he believes that Emmy believes that he (Nicolai) is unaware of it.

At each state, both mathematicians have two actions: work on it now or later. In Figure 2, we also depict the payoff matrix whose entries correspond to the story above.

The payoff matrices together with the information structure constitute a Bayesian game with unawareness. What could be a solution? An equilibrium should specify for each state an optimal strategy profile given the beliefs and awareness of the players at that state. We start

Figure 2: Information Structure and Payoffs in the Mathematician's Dilemma

ω_1

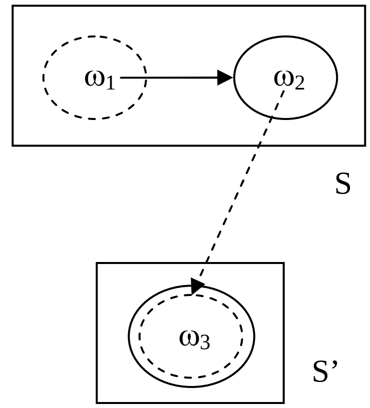
Emmy \ Nicolai	now	later
now	490, 490	1080, 0
later	0, 1080	590, 590

ω_2

Emmy \ Nicolai	now	later
now	990, -10	1080, 0
later	1000, 80	1090, 90

ω_3

Emmy \ Nicolai	now	later
now	-10, -10	80, 0
later	0, 80	90, 90



by considering optimal strategies at $\omega_3 \in S'$, the space where both players are unaware of the brilliant idea. In the symmetric game at ω_3, (later) is the dominant action for both players.

Next consider the state $\omega_2 \in S$. At this state, Emmy is aware of the brilliant idea because she believes $\omega_2 \in S$ with probability one. In contrast, Nicolai is unaware of it because at ω_2 he believes in $\omega_3 \in S'$ with probability one. Note that at ω_2 Emmy believes that Nicolai is unaware of it. Both player's dominant action is (later).

Finally, consider the state $\omega_1 \in S$. At this state both Emmy and Nicolai are aware of the brilliant idea since their "information sets" lie in S. But since Emmy's "information set" at ω_1 is $\{\omega_2\}$ and at ω_2 Nicolai's "information set " is $\{\omega_3\} \subset S'$, Emmy believes that Nicolai is unaware of it. Moreover, Nicolai believes that Emmy is aware of it, and he believes that Emmy believes that he (Nicolai) is unaware of it. So for Emmy the dominant action at ω_1 is later but for Nicolai it is now. That is, even though Emmy is aware of the brilliant idea and could solve the Riemann Hypothesis, she won't receive the desired award.

Note that the result of the example continues to hold if beliefs are slightly perturbed. E.g., at ω_1 and ω_2 Emmy could assign probability $\frac{1}{1000}$ to ω_1 and $\frac{999}{1000}$ to ω_2. □

4 Common Prior, Agreement, and Speculation

In this section, we define a common prior and show by example that the common prior assumption is too weak to rule out speculative trade under unawareness. With unawareness, we can have common certainty of willingness to trade but strict preference to trade. Yet, we are able to prove a "No-Trade" theorem according to which there can not be common certainty of strict preference to trade under unawareness. Moreover, we prove a "No-Agreeing-to-Disagree" theorem.

4.1 Common Belief

We define mutual and common belief as usual (e.g. Monderer and Samet, 1989):

From now on, we assume that the set of individuals I is at most countable.

Definition 9 *The mutual p-belief operator on events is defined by*

$$B^p(E) = \bigcap_{i \in I} B_i^p(E).$$

The common certainty operator on events is defined by

$$CB^1(E) = \bigcap_{n=1}^{\infty} \left(B^1\right)^n(E).$$

That is, the mutual p-belief of an event E is the event in which everybody p-believes the event E. Common certainty of E is the event that everybody is certain of the event E, and everybody is certain that everybody is certain of the event E, everybody is certain of that, ... ad infinitum. Common certainty is the generalization of common knowledge to the probabilistic notion of certainty. Note that Proposition 2 and the definition of the conjunction of events imply that $B^p(E)$ and $CB^1(E)$ are $S(E)$-based events, for any measurable event E.

We say that an event E is *common certainty* at $\omega \in \Omega$ if $\omega \in CB^1(E)$.

4.2 Priors and Common Priors

In a standard type space S, a *prior* P_i^S of player i is a convex combination of the beliefs of i's types in S (Samet, 1998). That is, for every event $E \in \mathcal{F}_S$,

$$P_i^S(E) = \int_S t_i(\cdot)(E)\, dP_i^S(\cdot). \tag{2}$$

In particular, if S is finite or countable, this equality holds if and only if

$$P_i^S(E) = \sum_{s \in S} t_i(s)(E)\, P_i^S(\{s\}). \tag{3}$$

In words, to find the probability $P_i^S(E)$ that the prior P_i^S assigns to an event E, one should check the beliefs $t_i(s)(E)$ ascribed by player i to the event E in each state $s \in S$, and then average these beliefs according to the weights $P_i^S(\{s\})$ assigned by the prior P_i^S to the different states $s \in S$.

P^S is a *common prior* on S if P^S is a prior for every player $i \in I$.

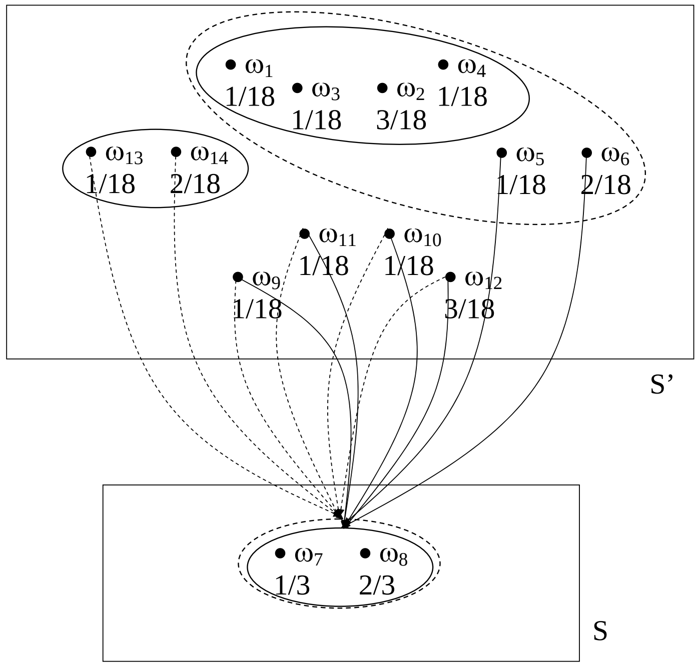
Here we generalize these definitions to unawareness structures, as follows.

Definition 10 (Prior) *A prior for player i is a system of probability measures $P_i = \left(P_i^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ such that*

1. *The system is projective: If $S' \preceq S$ then the marginal of P_i^S on S' is $P_i^{S'}$. (That is, if $E \in \Sigma$ is an event whose base-space $S(E)$ is lower or equal to S', then $P_i^S(E) = P_i^{S'}(E)$.)*

2. *Each probability measure P_i^S is a convex combination of i's beliefs in S: For every event $E \in \Sigma$ such that $S(E) \preceq S$,*

Figure 3: Illustration of a Common Prior



$$P_i^S\left(E \cap S \cap A_i\left(E\right)\right) = \int_{S \cap A_i(E)} t_i\left(\cdot\right)\left(E\right) dP_i^S\left(\cdot\right). \tag{2u}$$

$P = \left(P^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ *is a common prior if* P *is a prior for every player* $i \in I$.

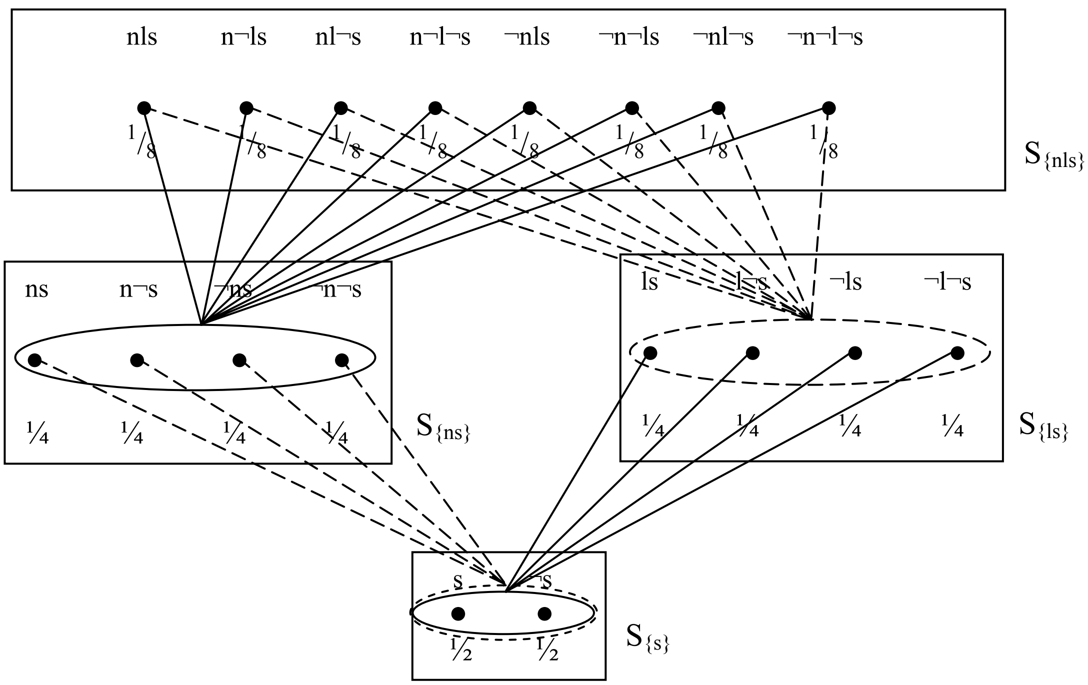
In particular, if S is finite or countable, equality (2u) holds if and only if

$$P_i^S\left(E \cap S \cap A_i\left(E\right)\right) = \sum_{s \in S \cap A_i(E)} t_i\left(s\right)\left(E\right) P_i^S\left(\{s\}\right). \tag{3u}$$

What is the reason for the difference between (2) and (2u) (or similarly between (3) and (3u))? With unawareness, $t_i\left(s\right)\left(E\right)$ is well defined only for states $s \in S$ in which player i is aware of E, i.e., the states $s \in S \cap A_i\left(E\right)$. Hence there is the difference in the definition of the domain of integration (or summation) on the right-hand side. Consequently, E (or equivalently $E \cap S$) on the left-hand side of (2) and (3) is replaced by $E \cap S \cap A_i\left(E\right)$ in (2u) and (3u).

Figure 3 illustrates a common prior in an unawareness belief structure. Odd (resp. even) states in the upper space project to the odd (resp. even) state in the lower space. There are two individuals, one indicated by the solid lines and ellipses and another by intermitted lines and ellipses. Note that the ratio of probabilities over odd and even states in each "information cell" coincides with the ratio in the "information cell" in the lower space.

Figure 4: Information Structure in the Speculative Trade Example



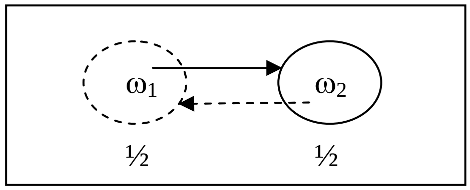
4.3 Speculation

In this section, we investigate whether the common prior assumption implies the absence of speculative trade (e.g. Milgrom and Stokey, 1982). The following example shows that speculation is possible under unawareness even if we assume that there is a common prior.

Example 2 (Speculative Trade under Unawareness) Consider the probabilistic version of the speculative trade example of Heifetz, Meier and Schipper (2006a). There is an owner, o, of a firm and a potential buyer, b, whose awareness differ. The owner is aware that there may be a costly lawsuit $[l]$ involving the firm, but he is unaware of a potential novelty $[n]$ enhancing the value of the firm. In contrast, the buyer is aware that there might be the innovation, but he is unaware of the lawsuit. Both are aware that the firm may face high sales $[s]$ or not in future.

The information structure is depicted in Figure 4. The solid lines and ellipses belong the buyer, whereas the intermitted lines and ellipses belong to the seller. At state (nls) the buyer's type has full support on space $S_{\{ns\}}$ whereas the seller's type has full support on space $S_{\{ls\}}$. Hence the buyer is uncertain whether the innovation obtains or not, and the seller is uncertain whether the lawsuit obtains. However, the buyer is certain that the seller is unaware of the innovation because the seller's type at states in $S_{\{ns\}}$ has full support on the space $S_{\{s\}}$. Similarly, the seller is certain that the buyer is unaware of the lawsuit.

Figure 5: Speculative Trade with Delusion



Suppose that the status quo value of the firm with high sales is 100 dollars, but only 80 dollars with low sales. If the potential innovation obtains, this would add 20 dollars to the value of the firm, whereas the potential lawsuit would cost the firm 20 dollars. The player's beliefs are stated in Figure 4 as well. According to these beliefs at state (nls), the buyer's expected value of the firm is 100, whereas the seller's expected value of the firm is 80 dollars. However, the buyer (resp. seller) is certain that the seller's (resp. buyer's) expected value is 90 dollars.

We assume that both players are rational in the sense of maximizing their respective payoff given their belief and awareness. The buyer (resp. seller) prefers to buy (resp. sell) at price x if her expected value of the firm is at least (resp. at most) x. The buyer (resp. seller) strictly prefers to buy (resp. sell) at price x if her expected value of the firm is strictly above (resp. strictly below) x.

Note that the beliefs stated below the states in each space are consistent with a common prior. However, at state (nls) and at the price 90 dollars, there is common certainty of preference to trade, but each player strictly prefers to trade. This is impossible in standard state-space structures with a common prior. $\square$

Despite this counter example to the "No-trade" theorems, we can prove below a generalized "No-trade" theorem according to which, if there is a common prior, then there can not be common certainty of strict preference to trade. That is, even with unawareness not "everything goes". We find this surprising, because unawareness can be interpreted as a special form of delusion: At a given state, a player may be certain of states in a very different lower state-space. The following example demonstrates that speculative trade is possible in delusional standard state-space structures with a common prior.

Example 3 (Speculative Trade with Delusion) Consider the information structure in Figure 5. The common prior and the information structure allows the dashed player to have a posterior of $t_{dashed}(\omega_1)(\{\omega_1\}) = t_{dashed}(\omega_2)(\{\omega_1\}) = 1$ and the solid player $t_{solid}(\omega_1)(\{\omega_2\}) = t_{solid}(\omega_2)(\{\omega_2\}) = 1$. So they may happily disagree on the expected value of a random variable defined on this standard state-space. $\square$

Definition 11 *A common prior* $P = \left(P^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ *is* non-degenerate *if and only if for all* $i \in I$ *and* $\omega \in \Omega$*: If* $t_i(\omega) \in \triangle(S')$*, for some* S'*, then* $[t_i(\omega)] \cap S' \in \mathcal{F}_{S'}$ *and*

$P^S \left(([t_i(\omega)] \cap S')^{\uparrow} \cap S \right) > 0$ *for all* $S \succeq S'$.

For every type, a non-degenerate common prior puts positive weight on the set of "stationary" states where the player has this type. This condition implies that for each player there can be at most countably many types in each space.

Definition 12 *Let x_1 and x_2 be real numbers and v a random variable on Ω. Define the sets* $E_1^{\leq x_1} := \left\{ \omega \in \Omega : \int_{S_{t_1(\omega)}} v(\cdot)\, d(t_1(\omega))(\cdot) \leq x_1 \right\}$ *and* $E_2^{\geq x_2} := \left\{ \omega \in \Omega : \int_{S_{t_2(\omega)}} v(\cdot)\, d(t_2(\omega))(\cdot) \geq x_2 \right\}$. *We say that at ω, conditional on his information, player 1 (resp. player 2) believes that the expectation of v is weakly below x_1 (resp. weakly above x_2) if and only if $\omega \in E_1^{\leq x_1}$ (resp. $\omega \in E_1^{\geq x_2}$).*

Note that the sets $E_1^{\leq x_1}$ or $E_2^{\geq x_2}$ may not be events in our unawareness belief structure, because $v(\omega) \neq v(\omega_S)$ is allowed, for $\omega \in S' \succ S$. Yet, we can define p-belief, mutual p-belief and common certainty for measurable[7] subsets of Ω, and show that the properties stated in Proposition 7 and 9 obtain as well (see Meier and Schipper, 2007).[8]

Theorem 1 *Let $\underline{\Omega}$ be a finite unawareness belief structure and $P = \left(P^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ be a non-degenerate common prior. Then there is no state $\tilde{\omega} \in \Omega$ such that there are a random variable $v : \Omega \longrightarrow \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$, with the following property: at $\tilde{\omega}$ it is common certainty that conditional on her information, player 1 believes that the expectation of v is weakly below x_1 and, conditional on his information, player 2 believes that the expectation of v is weakly above x_2.*

The theorem says that if there is a non-degenerate common prior, then there can not be common certainty of strict preference to trade. Together with our example of speculative trade under unawareness we conclude that a common prior does not rule out speculation under unawareness but it can never be common certainty that both players expect to strictly gain from speculation. The theorem implies immediately as a corollary that given a non-degenerate common prior, *arbitrary small transaction fees rule out speculative trade under unawareness.*

So, with respect to speculative trade, heterogeneous unawareness with a common prior is "intermediate" between common awareness with heterogeneous priors on the one hand, and common awareness with a common prior on the other hand. With heterogeneous priors even in standard state-spaces, common certainty of strict preference to trade is possible.

In a separate appendix to this paper, Meier and Schipper (2007), we extend the above "No-trade" theorem to infinite unawareness belief structures. To this end we introduce a topological unawareness belief structure and consider as a technical device a "flattened" structure with the union of all spaces in the lattice as the state-space. All properties of p-belief in Proposition 7 and 9 are extended to measurable subsets of Ω.

[7] A subset E of the union of the state spaces is defined to be measurable if and only if the intersection $E \cap S$ is measurable in S, for every state space S.

[8] Contrary to our definition of the negation of an event, in point (ii) of Proposition 7, $\neg E$ is here understood to be the relative complement of E with respect to the union of state spaces.

4.4 Agreement

For an event E and $p \in [0,1]$ define the set $[t_i(E) = p] := \{\omega \in \Omega : t_i(\omega)(E) = p\}$, if $\{\omega \in \Omega : t_i(\omega)(E) = p\}$ is nonempty, and otherwise set $[t_i(E) = p] := \emptyset^{S(E)}$.

Lemma 1 *$[t_i(E) = p]$ is a $S(E)$-based event.*

PROOF. $[t_i(E) = p] = B_i^p(E) \cap B_i^{1-p}(\neg E)$. Hence the proof follows from Proposition 2. $\square$

The following proposition is a generalization of the standard *"No-Agreeing-to-Disagree" theorem* (Aumann, 1976):

Proposition 6 *Let $\underline{\Omega}$ be an unawareness belief structure, G be an event and $p_i \in [0,1]$, for $i \in I$. Suppose there exists a common prior $P = \left(P^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ such that for some space $S \succeq S(G)$ we have $P^S(CB^1(\bigcap_{i \in I}[t_i(G) = p_i])) > 0$. Then $p_i = p_j$, for all $i, j \in I$.*[9]

The proposition says the following: Suppose individuals have a common prior that is weakly non-degenerate in the sense that it assigns strict positive probability to the event that posteriors of G are common certainty. Then common certainty of posteriors for the event G implies that those posteriors must agree across all individuals. So individuals with a common prior can not agree-to-disagree on the posteriors of events which they are all aware of.

Note, that a non-degenerate common prior (Definition 11) implies the condition $P^S(CB^1(\bigcap_{i \in I}[t_i(G) = p_i])) > 0$ in Proposition 6 if $CB^1(\bigcap_{i \in I}[t_i(G) = p_i])$ is nonempty and $S \succeq S(G)$.

Appendices

A Properties of Belief and Awareness

Proposition 7 *Let E and F be events, $\{E_l\}_{l=1,2,\ldots}$ be an at most countable collection of events, and $p, q \in [0,1]$. The following properties of belief obtain:*

- *(o)* $B_i^p(E) \subseteq B_i^q(E)$, *for* $q \leq p$,
- *(i) Necessitation:* $B_i^1(\Omega) = \Omega$,
- *(ii) Additivity:* $B_i^p(E) \subseteq \neg B_i^q(\neg E)$, *for* $p + q > 1$,
- *(iiia)* $B_i^p\left(\bigcap_{l=1}^{\infty} E_l\right) \subseteq \bigcap_{l=1}^{\infty} B_i^p(E_l)$,
- *(iiib) for any decreasing sequence of events* $\{E_l\}_{l=1}^{\infty}$, $B_i^p\left(\bigcap_{l=1}^{\infty} E_l\right) = \bigcap_{l=1}^{\infty} B_i^p(E_l)$,
- *(iiic)* $B_i^1\left(\bigcap_{l=1}^{\infty} E_l\right) = \bigcap_{l=1}^{\infty} B_i^1(E_l)$,
- *(iv) Monotonicity:* $E \subseteq F$ *implies* $B_i^p(E) \subseteq B_i^p(F)$,
- *(v) Introspection:* $B_i^p(E) \subseteq B_i^1 B_i^p(E)$.

[9] In the appendix, we prove a more general version in which we require only a "local" common prior on a space $S \succeq S(G)$ satisfying the condition stated in the proposition.

In our unawareness belief structure, Necessitation means that an individual always is certain of the universal event Ω, i.e. she is certain of "tautologies with the lowest expressive power." (ii) means that if an individual believes an event E with at least probability p, then she can not believe the negation of E with any probability strictly greater than $1-p$. Property (iii a - c) are variations of conjunction, i.e., if an individual believes a conjunction of events with probability at least p, then she p-believes each of the events. The interpretation of monotonicity is: If an event E implies an event F, then p-believing the event E implies that the individual also p-believes the event F. Property (v) concerns the introspection of belief: If an individual believes the event E with at least probability p then she is certain that she believes the event E with at least probability p.

Proposition 8 *Let E be an event and $p, q \in [0,1]$. The following properties of awareness and belief obtain:*

1. *Plausibility:* $U_i(E) \subseteq \neg B_i^p(E) \cap \neg B_i^p \neg B_i^p(E)$,
2. *Strong Plausibility:* $U_i(E) \subseteq \bigcap_{n=1}^{\infty} \left(\neg B_i^p\right)^n (E)$,
3. *B^pU Introspection:* $\begin{array}{lcll} B_i^p U_i(E) & = & \emptyset^{S(E)} & \text{for } p \in (0,1] \\ B_i^0 U_i(E) & = & A_i(E) & \end{array}$,
4. *AU Introspection:* $U_i(E) = U_i U_i(E)$,
5. *Weak Necessitation:* $A_i(E) = B_i^1\left(S(E)^{\uparrow}\right)$,
6. $\begin{array}{lcl} B_i^p(E) & \subseteq & A_i(E) \\ B_i^0(E) & = & A_i(E) \end{array}$,
7. $B_i^p(E) \subseteq A_i B_i^q(E)$,
8. *Symmetry:* $A_i(E) = A_i(\neg E)$,
9. *A Conjunction:* $\bigcap_{\lambda \in L} A_i(E_\lambda) = A_i\left(\bigcap_{\lambda \in L} E_\lambda\right)$,
10. *AB^p Self Reflection:* $A_i B_i^p(E) = A_i(E)$,
11. *AA Self Reflection:* $A_i A_i(E) = A_i(E)$,
12. $B_i^p A_i(E) = A_i(E)$.

These properties are analogous to the properties in unawareness knowledge structures (Heifetz, Meier and Schipper, 2006a, 2006b). In the context of knowledge, Properties 1 to 5 have been suggested by Dekel, Lipman and Rustichini (1998), and 8 to 11 by Fagin and Halpern (1988), Modica and Rustichini (1999) and Halpern (2001).

Note that properties 3, 4, 5, 8, 9, 11, and 12 hold also for non-measurable events, because even if E is not measurable, by 5. $A_i(E)$ is measurable.

Property 6 implies that probability zero belief is distinct from unawareness. In fact, an individual is aware of an event if and only if she assigns at least probability zero to this event.

Although we model awareness of events, Property 8 suggests that we model a notion of awareness of issues or questions. Let an issue or question (E.g., is the stock market crashing?) be such that it can be answered with in the affirmative (The stock market is crashing.) or the negative (The stock market is not crashing.). By symmetry (Property 8), an individual is aware of an event if and only if she is aware of the its negation. Thus we model the awareness of questions and issues rather than just single events. In fact, by weak necessitation, Property 5, an individual is aware of an event E if and only if she is aware of any event that can be "expressed" in a space with the same "expressive power" as the base-space of E.

Definition 13 *An event E is evident if for each $i \in I$, $E \subseteq B_i^1(E)$.*

Proposition 9 *For every event $F \in \Sigma$:*

(i) $CB^1(F)$ is evident, that is $CB^1(F) \subseteq B_i^1(CB^1(F))$ for all $i \in I$.

(ii) There exists an evident event E such that $\omega \in E$ and $E \subseteq B_i^1(F)$ for all $i \in I$, if and only if $\omega \in CB^1(F)$.

The proof is analogous to Proposition 3 in Monderer and Samet (1989) for a standard state-space and thus omitted.

Analogously to mutual belief and common belief, we define mutual awareness and common awareness:

Definition 14 *The mutual awareness operator on events is defined by*

$$A(E) = \bigcap_{i \in I} A_i(E),$$

and the common awareness operator on events is defined by

$$CA(E) = \bigcap_{n=1}^{\infty} (A)^n (E).$$

Mutual awareness of an event E is the event that everybody is aware of E. Common awareness of an event E is the event that everybody is aware of E, everybody is aware that everybody is aware of E, everybody is aware of that ... ad infinitum.

Proposition 10 *Let E be an event and $p, q \in [0, 1]$. The following multi-person properties obtain:*

1. $A_i(E) = A_i A_j(E)$,
2. $A_i(E) = A_i B_j^p(E)$,
3. $B_i^p(E) \subseteq A_i B_j^q(E)$,
4. $B_i^p(E) \subseteq A_i A_j(E)$,
5. $CA(E) = A(E)$,
6. $CB^1(E) \subseteq CA(E)$,
7. $B^p(E) \subseteq A(E)$,
   $B^0(E) = A(E)$,
8. $B^p(E) \subseteq CA(E)$,
   $B^0(E) = CA(E)$,
9. $A(E) = B^1(S(E)^{\uparrow})$,
10. $CA(E) = B^1(S(E)^{\uparrow})$,
11. $CB^1(S(E)^{\uparrow}) \subseteq A(E)$,
12. $CB^1(S(E)^{\uparrow}) \subseteq CA(E)$.

Note that properties 1, 5, 9, 10, 11 and 12 also hold for non-measurable events.

B Generalized Bayesian Games with Unawareness

B.1 Allowing for Unawareness of Actions

Bayesian games with unawareness of events in Definition 6 do not allow us to model properly unawareness of actions. In standard Bayesian game theory, ignorance of actions is modelled by the assumption that players will never use such actions, because payoffs are extremely low (i.e. highly negative) (see the discussion in Harsanyi, 1967, p. 168). We do not follow this convention here. Even in standard Bayesian games this convention is questionable, because it applies only to rational players. If there is lack of common belief of rationality then a player's type being ignorant of an action is indeed different from her obtaining a very low payoff from playing this action (see Hu and Stuart, 2001, for a discussion). In this subsection we introduce unawareness of actions and discuss the notion of strategy in Bayesian games with unawareness.

Definition 15 *A Bayesian game with unawareness of events and actions consists of a unawareness belief structure* $\left\langle \mathcal{S}, \left(r^{S_\alpha}_{S_\beta}\right)_{S_\beta \preceq S_\alpha}, (t_i)_{i\in I} \right\rangle$ *that is augmented by a tuple* $\left\langle (M_i)_{i\in I}, (\mathcal{M}_i)_{i\in I}, (u_i)_{i\in I} \right\rangle$ *defined as follows: For each player* $i \in I$*, there is*

(i) a finite nonempty set of actions M_i *and a correspondence* $\mathcal{M}_i : \Omega \longrightarrow 2^{M_i} \setminus \{\emptyset\}$ *such that for any* $M_i' \subseteq M_i$, $[M_i'] := \{\omega \in \Omega : M_i' \subseteq \mathcal{M}_i(\omega)\}$ *is an event (in the sense of the unawareness belief structure), and* $\omega', \omega'' \in [t_i(\omega)] \cap S_{t_i(\omega)}$ *implies* $\mathcal{M}_i(\omega') = \mathcal{M}_i(\omega'')$*, for all* $\omega \in \Omega$*,*

(ii) a utility function $u_i : \bigcup_{\omega\in\Omega} \left(\prod_{i\in I} \mathcal{M}_i(\omega)\right) \times \{\omega\} \longrightarrow \mathbb{R}$*.*

This definition allows for unawareness of events as well as actions. Which actions a player i has available at what state is described explicitly by the correspondence $\mathcal{M}_i$. Any set of available actions is associated with an event in our unawareness belief structure. We require that, for each type of each player, the sets of available actions are identical across states in the space on which this type is defined and at which the player's type coincides with this type. Note, that if $\omega \notin S_{t_i(\omega)}$, then it is allowed that $\mathcal{M}_i(\omega')$ is a proper subset of $\mathcal{M}_i(\omega)$, for $\omega' \in [t_i(\omega)] \cap S_{t_i(\omega)}$. This allows in addition to unawareness of other players' actions also for unawareness of own actions. Note, that we exclude that at a state, a player considers it possible that she has an action available, which, in fact, is not available to her in this state. This is to avoid the following conceptional problem: What should happen if a player is to take an action that is not available to her?

B.2 Allowing for Unawareness of Players

So far, we did not allow for unawareness of players. In standard Bayesian game theory, ignorance of players is modelled by dummy players, i.e., players that obtain a very low payoff for all actions except one (dummy) action. This is distinct from being unable to conceive of a player at all. In this subsection we allow for unawareness of players. This requires that we generalize our interactive unawareness belief structure such that a player may exists only at some states but not at others.

Definition 16 *A Bayesian game with unawareness is a tuple*

$$\Gamma(\underline{\Omega}) := \left\langle \mathcal{S}, \left(r^{S_\alpha}_{S_\beta}\right)_{S_\beta \preceq S_\alpha}, \mathcal{E}, (t_i)_{i\in I}, (M_i)_{i\in I}, (\mathcal{M}_i)_{i\in I}, (u_i)_{i\in I} \right\rangle$$

defined as follows:

(0) $\mathcal{S} = \{S_\alpha\}_{\alpha\in\mathcal{A}}$ *is as before a complete lattice of spaces with surjective projections* $(r^{S_\alpha}_{S_\beta})$*, for* $S_\beta \preceq S_\alpha$ *(see section 2).*

(i) $\mathcal{E} : I \longrightarrow \Sigma$ *is the "existence" correspondence that assigns to each player* $i \in I$ *an event in which she exists.*

For each player $i \in I$*:*

(ii) $t_i : \mathcal{E}(i) \longrightarrow \bigcup_{S \in \mathcal{S}_i} \Delta(S)$ *is a type mapping that satisfies Properties (0) to (3) (see section 2) such that for every* $\omega \in \mathcal{E}(i)$, $t_i(\omega)(\mathcal{E}(i)) = 1$. $\mathcal{S}_i := \{S \in \mathcal{S} : \mathcal{E}(i) \cap S \neq \emptyset\}$ *is the complete sublattice of spaces with states in which player* i *exists.*

(iii) M_i *is a finite nonempty set of actions and* $\mathcal{M}_i : \mathcal{E}(i) \longrightarrow 2^{M_i} \setminus \{\emptyset\}$ *is a correspondence such that for any* $M_i' \subseteq M_i$, $[M_i'] := \{\omega \in \mathcal{E}(i) : M_i' \subseteq \mathcal{M}_i(\omega)\}$ *is an event (in the sense of the unawareness belief structure), and* $\omega', \omega'' \in S_{t_i(\omega)} \cap [t_i(\omega)]$ *implies* $\mathcal{M}_i(\omega') = \mathcal{M}_i(\omega'')$*, for all* $\omega \in \mathcal{E}(i)$*,*

(vi) $u_i : \bigcup_{\omega \in \mathcal{E}(i)} \left(\prod_{i \in I(\omega)} \mathcal{M}_i(\omega)\right) \times \{\omega\} \longrightarrow \mathbb{R}$ *is a utility function, where* $I(\omega) := \{i \in I : \omega \in \mathcal{E}(i)\}$*.*

This game allows for unawareness of events, actions and players. For every player $i \in I$, the "existence" correspondence $\mathcal{E}$ assigns to i the event in which she exists. Consequently we restrict player i's type mapping to states at which she exists. Moreover, player i's type is concentrated only on states in which she exists. A player can not assign strict positive probability to states at which she does not exist. The domain of the correspondence $\mathcal{M}_i$, that assigns to states a non-empty set of actions for player i, is also restricted to the set of states in which player i exists. We do not allow a player to have some actions in states in which she does not exist. The dimension of the domain of a utility function may vary from state to state, since players may exists in some states but not in others, and each players utility at a state depends on the actions of all the players that exist in that state.

Note that if $\mathcal{E}(i) = \Omega$ for all $i \in I$, then we obtain a unawareness belief structure and a Bayesian game with unawareness of events and actions as defined before.

Note further that if $\omega \in \mathcal{E}(i)$, then $[t_i(\omega)] := \{\omega' \in \Omega : t_i(\omega') = t_i(\omega)\} \subseteq \mathcal{E}(i)$.

B.3 Equilibrium

By allowing also for unawareness of actions and players, we need to adapt slightly the definition of a strategy:

Definition 17 *A strategy of player* i *is a function* $\sigma_i : \mathcal{E}(i) \longrightarrow \Delta(M_i)$ *such that for all* $\omega \in \mathcal{E}(i)$*,*

(i) $\sigma_i(\omega) \in \Delta(\mathcal{M}_i(\omega_{S_{t_i(\omega)}}))$*, and*

(ii) $t_i(\omega') = t_i(\omega)$ *implies* $\sigma_i(\omega') = \sigma_i(\omega)$*.*

Denote $\sigma_{S_{t_i(\omega)}} := \left((\sigma_j(\omega'))_{j \in I(\omega')}\right)_{\omega' \in S_{t_i(\omega)}}$. The *expected utility* of player-type $(i, t_i(\omega))$ from the strategy profile $\sigma_{S_{t_i(\omega)}}$ is given by

$$U_{(i,t_i(\omega))}(\sigma_{S_{t_i(\omega)}}) := \int_{\omega' \in S_{t_i(\omega)}} \sum_{m \in \prod_{j \in I(\omega')} \mathcal{M}_j(\omega'_{S_{t_j(\omega')}})} \prod_{j \in I(\omega')} \sigma_j(\omega')(m_j) \cdot u_i\left((m_j)_{j \in I(\omega')}, \omega'\right) dt_i(\omega)(\omega'). \tag{4}$$

Definition 18 (Equilibrium) *An* equilibrium *of a Bayesian game with unawareness* $\Gamma(\underline{\Omega})$ *is a Nash equilibrium of the strategic game defined by:*

(i) $\{(i, t_i(\omega)) : \omega \in \Omega \text{ and } i \in I(\omega)\}$ *is the set of players,*

and for each player $(i, t_i(\omega))$*,*

(ii) the set of mixed strategies is $\Delta(\mathcal{M}_i(\omega_{S_{t_i(\omega)}}))$*, and*

(iii) the utility function is given by equation (4).

Proposition 3 generalizes to Bayesian games with unawareness. If I, Ω, and $(M_i)_{i\in I}$ are finite, then, by Nash's Theorem, there exists an equilibrium.

Definition 19 *Given a Bayesian game with unawareness* $\Gamma(\underline{\Omega}) = \left\langle \mathcal{S}, \left(r_{S_\beta}^{S_\alpha}\right)_{S_\beta \preceq S_\alpha}, \mathcal{E}, (t_i)_{i\in I}, (M_i)_{i\in I}, (\mathcal{M}_i)_{i\in I}, (u_i)_{i\in I} \right\rangle$ *we can define an* S'*-partial Bayesian game with unawareness* $\Gamma(\underline{\Omega}') = \left\langle l(S'), \left(r_{S_\beta}^{S_\alpha}\right)_{S_\beta \preceq S_\alpha}, \mathcal{E}', (t_i)_{i\in I}, (M_i)_{i\in I}, (\mathcal{M}_i)_{i\in I}, (u_i)_{i\in I} \right\rangle$ *in which* $\left(r_{S_\beta}^{S_\alpha}\right)$ *are restricted to* $S_\alpha, S_\beta \in l(S')$*,* $\mathcal{E}'(i) = \mathcal{E}(i) \cap \Omega'$*, where* $\Omega' = \bigcup_{S\in l(S')} S$*, and for any* $i \in I$ *the domain of* $\mathcal{M}_i$ *is restricted to* $\mathcal{E}'(i)$*.*

Propositions 4 and 5 generalize to Bayesian games with unawareness. In fact, the proofs in the appendix are stated for this more general setting.

Example 4 (Feinberg, 2005) The following interesting game due to Feinberg (2005) is an example of unawareness of actions. It allows us also to compare our unawareness belief structures with the work by Feinberg (2005). Consider the strategic 3x3 game

		Burkhard		
		b_1	b_2	b_3
	a_1	$(0,2)$	$(3,3)$	$(0,2)$
Amanda	a_2	$(2,2)$	$(2,1)$	$(2,1)$
	a_3	$(1,0)$	$(4,0)$	$(0,1)$

This game has a unique dominance solvable Nash equilibrium, (a_2, b_1). Consider now a game with unawareness: The set of players remains unchanged, Amanda, A, and Burkhard, B. There are two state-spaces, S and S' with $S \succ S'$. In particular, $S = \{\omega_1, \omega_2\}$ and $S' = \{\omega_3\}$. The information structure is given by the type mappings

$$t_A(\omega_1)(\{w_2\}) = t_A(\omega_2)(\{w_2\}) = t_A(\omega_3)(\{\omega_3\}) = 1,$$

$$t_B(\omega_1)(\{\omega_1\}) = t_B(\omega_2)(\{\omega_3\}) = t_B(\omega_3)(\{\omega_3\}) = 1.$$

Actions are specified by

$$\mathcal{M}_A(\omega_1) = \mathcal{M}_A(\omega_2) = \{a_1, a_2, a_3\}, \mathcal{M}_A(\omega_3) = \{a_1, a_2\},$$

$$\mathcal{M}_B(\omega_1) = \mathcal{M}_B(\omega_2) = \mathcal{M}_B(\omega_3) = \{b_1, b_2, b_3\}.$$

The information structure is the same as in the Mathematician's Dilemma (with Emmy being now Amanda and Nicolai being now Burkhard) in Figure 2. At states ω_1 and ω_2, payoffs are given by the above payoff matrix. At state ω_3, payoffs are given by the sub-matrix spanned by rows a_1 and a_2 and columns b_1, b_2, and b_3 in the above matrix, i.e.,

		Burkhard		
		b_1	b_2	b_3
Amanda	a_1	$(0,2)$	$(3,3)$	$(0,2)$
	a_2	$(2,2)$	$(2,1)$	$(2,1)$

We claim that

$$(\sigma_A(\omega), \sigma_B(\omega)) = \begin{cases} (a_3, b_3) & \text{if} \quad \omega = \omega_1 \\ (a_3, b_2) & \text{if} \quad \omega = \omega_2 \\ (a_1, b_2) & \text{if} \quad \omega = \omega_3 \end{cases}$$

is an equilibrium. To see this, note that the game at ω_3 has two pure equilibria, (a_2, b_1) and (a_1, b_2) in the S'-partial game, where the latter is payoff dominant. At ω_3, both players are unaware of action a_3 in ω_3. The unique dominance solvable Nash equilibrium (a_2, b_1) of the original game (without unawareness of actions) remains an equilibrium because none of the players is unaware of an equilibrium action and equilibrium actions remain best responses if some other actions are deleted. Moreover, after deleting action a_3 (the action both players are unaware of at state ω_3 in S'), the game has another Nash equilibrium (a_1, b_2). At ω_1, both players are aware of all actions but Amanda believes that Burkhard is unaware of action a_3. Hence Amanda believes that Burkhard thinks that (a_1, b_2) is a Nash equilibrium. Amanda's best response to Burkhard playing b_2 is a_3. Moreover, since at ω_1 Burkhard is aware of all actions and he believes that Amanda believes that Burkhard is unaware of action a_3, his best response to Amanda playing a_3 is b_3. Note that in this equilibrium at ω_1, both receive a low payoff (compared to the Nash equilibria discussed previously).

Feinberg (2005) obtains (a_3, b_3) as an equilibrium if both players are aware of all actions, Amanda is 'unaware' that Burkhard is aware of all of her actions, and Burkhard is 'aware' that Amanda is 'aware' of Burkhard being unaware of a_3.[10] That is, in Feinberg (2005) a player can be aware of an event but unaware that somebody else is aware of it. This is in contrast to our unawareness belief structure, where according to Proposition 10, 1., a player is aware of an event if and only if she is aware that somebody else could be aware of it. That is, if a player can reason about some issue then she can also reason that somebody else can reason about that issue. We obtain (a_3, b_3) as an equilibrium if both players are aware of all actions, Amanda does not believe that Burkhard is aware of a_3, Burkhard believes that Amanda believes that Burkhard is unaware of a_3. The example suggests, that higher order 'awareness' in Feinberg (2005) operates like belief in our unawareness belief structure. Note however, that Feinberg (2005) does not define a notion of belief in his framework. $\square$

C Proofs

C.1 Proof of Proposition 1

$A_i(E)$ is an $S(E)$-based event if there exists a subset $D \subseteq S(E)$ s.t. $D^{\uparrow} = A_i(E)$.

Assume that $A_i(E)$ is non-empty. Define $D := \{\omega \in S(E) : t_i(\omega) \in \Delta(S(E))\}$. By definition of the awareness operator, $D = A_i(E) \cap S(E)$. We show that $D^{\uparrow} = A_i(E)$.

Let $\omega \in D^{\uparrow}$, that is $\omega \in S'$ for some $S' \succeq S(E)$ and $\omega_{S(E)} \in D$. This is equivalent to $t_i(\omega_{S(E)}) \in \Delta(S(E))$. By 0. follows $S' \succeq S_{t_i(\omega)}$. By 3. we have $S_{t_i(\omega)} \succeq S(E)$. Thus $\omega \in A_i(E)$. (Note that $A_i(E) = \{\omega \in \Omega : S_{t_i(\omega)} \succeq S(E)\}$.)

In the reverse direction, let $\omega \in A_i(E)$, i.e., $t_i(\omega) \in \Delta(S)$ with $S \succeq S(E)$. By 0., $\omega \in S'$ with $S' \succeq S$. Consider $\omega_{S(E)}$. By 2., $t_i(\omega_{S(E)}) = t_i(\omega)_{|S(E)}$. Hence $\omega_{S(E)} \in D$. Thus $\omega \in D^{\uparrow}$.

Finally, if $A_i(E)$ is empty, then by definition of the awareness operator, we have $A_i(E) = \emptyset^{S(E)}$. $\square$

C.2 Proof of Proposition 2

$B_i^p(E)$ is an $S(E)$-based event if there exists a subset $D \subseteq S(E)$ s.t. $D^{\uparrow} = B_i^p(E)$. Assume that $B_i^p(E)$ is non-empty. Define $D := \{\omega \in S(E) : t_i(\omega)(E) \geq p\}$. By definition of the p-belief operator,

[10] When writing '...', we indicate that those notions differ from our notions used in this paper.

$D = B_i^p(E) \cap S(E)$. We show that $D^{\uparrow} = B_i^p(E)$.

Let $\omega \in D^{\uparrow}$, that is $\omega \in S'$ for some $S' \succeq S(E)$ and $\omega_{S(E)} \in D$. This is equivalent to $t_i(\omega_{S(E)})(E) \geq p$. By 0. $S_{t_i(\omega_{S(E)})} = S(E)$. By 3. we have $S_{t_i(\omega)} \succeq S(E)$. By 2. it follows that $p \leq t_i(\omega_{S(E)})(E) = t_i(\omega)_{|S(E)}(E)$. Hence $t_i(\omega)(E) \geq p$. Thus $\omega \in B_i^p(E)$.

In the reverse direction, let $\omega \in B_i^p(E)$, i.e., $t_i(\omega)(E) \geq p$. Since $t_i(\omega)(E) \geq p$ it follows that $S_{t_i(\omega)} \succeq S(E)$. Let $\omega \in S'$. By 0. $S' \succeq S_{t_i(\omega)}$. Consider $\omega_{S(E)}$. By 2., $t_i(\omega_{S(E)})(E) = t_i(\omega)(E)_{|S(E)} \geq p$. Hence $\omega_{S(E)} \in D$. Thus $\omega \in D^{\uparrow}$.

Finally, if $B_i^p(E)$ is empty, then by definition of the p-belief operator, we have $B_i^p(E) = \emptyset^{S(E)}$. $\square$

C.3 Proof of Remark 1

Define $D := \{\omega' \in S_{t_i(\omega)} : t_i(\omega') = t_i(\omega)\}$. I.e., $D = Ben_i(\omega) \cap S_{t_i(\omega)}$. We need to show that $D^{\uparrow} = Ben_i(\omega)$.

Consider first "$\subseteq$": If $\omega' \in D^{\uparrow}$ then $\omega'_{S_{t_i(\omega)}} \in Ben_i(\omega)$. This is equivalent to $t_i(\omega'_{S_{t_i(\omega)}}) = t_i(\omega) \in \triangle(S_{t_i(\omega)})$. By (3) we have $S_{t_i(\omega')} \succeq S_{t_i(\omega)}$. By (2), $t_i(\omega'_{S_{t_i(\omega)}}) = t_i(\omega')_{|S_{t_i(\omega)}}$. It follows that $t_i(\omega')_{|S_{t_i(\omega)}} = t_i(\omega)$. Thus $\omega' \in Ben_i(\omega)$.

"$\supseteq$": $\omega' \in Ben_i(\omega)$ if and only if $t_i(\omega')_{|S_{t_i(\omega)}} = t_i(\omega)$. Hence for $\omega' \in Ben_i(\omega)$, we have $S_{t_i(\omega')} \succeq S_{t_i(\omega)}$. By (2) $t_i(\omega'_{S_{t_i(\omega)}}) = t_i(\omega')_{|S_{t_i(\omega)}} = t_i(\omega)$. Hence $\omega'_{S_{t_i(\omega)}} \in D$. Thus $\omega' \in D^{\uparrow}$. $\square$

C.4 Proof of Proposition 7

(0) $B_i^p(E) \subseteq B_i^q(E)$ for $p, q \in [0, 1]$ with $q \leq p$ is trivial.

(i) $B_i^1(\Omega) \subseteq \Omega$ holds trivially. In the reverse direction, note that $t_i(\omega)(\Omega) = t_i(\omega)(\Omega \cap S_{t_i(\omega)}) = t_i(\omega)(S_{t_i(\omega)}) = 1$ for all $\omega \in \Omega$. Thus $\Omega \subseteq B_i^1(\Omega)$.

(ii) $\omega \in B_i^p(E)$ if and only if $t_i(\omega)(E) \geq p$. Since $t_i(\omega)$ is an additive probability measure, $t_i(\omega)(\neg E) \leq 1 - p$. Hence $\omega \in \neg B_i^q(\neg E)$ for $q > 1 - p$.

(iiia) $\omega \in B_i^p\left(\bigcap_{l=1}^{\infty} E_l\right)$ if and only if $t_i(\omega)\left(\bigcap_{l=1}^{\infty} E_l\right) \geq p$. Monotonicity of the probability measure $t_i(\omega)$ implies $t_i(\omega)(E_l) \geq p$ for all $l = 1, 2, ...$, which is equivalent to $\omega \in \bigcap_{l=1}^{\infty} B_i^p(E_l)$.

(iiib) It is enough to show that any sequence of events $\{E_l\}_{l=1}^{\infty}$ with $E_l \supseteq E_{l+1}$ for $l = 1, 2, ...$ we have $B_i^p\left(\bigcap_{l=1}^{\infty} E_l\right) \supseteq \bigcap_{l=1}^{\infty} B_i^p(E_l)$. $\omega \in \bigcap_{l=1}^{\infty} B_i^p(E_l)$ if and only if $t_i(\omega)(E_l) \geq p$ for $l = 1, 2, ...$ Since $t_i(\omega)$ is a countable additive probability measure, it is continuous from above. That is, if $E_l \supseteq E_{l+1}$ for $l = 1, 2, ...$, we have $\lim_{l \to \infty} t_i(\omega)(E_l) = t_i(\omega)\left(\bigcap_{l=1}^{\infty} E_l\right)$. Since for every $l = 1, 2, ...$, $t_i(\omega)(E_l) \geq p$, we have $p \leq \lim_{l \to \infty} t_i(\omega)(E_l) = t_i(\omega)\left(\bigcap_{l=1}^{\infty} E_l\right)$. Hence $\omega \in B_i^p\left(\bigcap_{l=1}^{\infty} E_l\right)$.

(iiic) It is enough to show that $B_i^1\left(\bigcap_{l=1}^{\infty} E_l\right) \supseteq \bigcap_{l=1}^{\infty} B_i^1(E_l)$. $\omega \in \bigcap_{l=1}^{\infty} B_i^1(E_l)$ if and only if $t_i(\omega)(E_l) = 1$ for $l = 1, 2, ...$ Since $t_i(\omega)$ is a countable additive probability measure, it satisfies Bonferroni's Inequality. I.e., $t_i(\omega)\left(\bigcap_{l=1}^{\infty} E_l\right) \geq 1 - \sum_{l=1}^{\infty} 1 - t_i(\omega)(E_l)$. Since $t_i(\omega)(E_l) = 1$ for all $l = 1, 2, ...$, we have $1 - t_i(\omega)(E_l) = 0$ for all $l = 1, 2, ...$, and hence $\sum_{l=1}^{\infty} 1 - t_i(\omega)(E_l) = 0$. It follows that $t_i(\omega)\left(\bigcap_{l=1}^{\infty} E_l\right) = 1$. We conclude that $\omega \in B_i^1\left(\bigcap_{l=1}^{\infty} E_l\right)$.

(iv) Since $t_i(\omega)$ is a probability measure (satisfying monotonicity) for any $\omega \in \Omega$, $E \subseteq F$ implies that if $t_i(\omega)(E) \geq p$ then $t_i(\omega)(F) \geq p$.

(v) Let $\omega \in B_i^p(E)$. Then $t_i(\omega)(E) \geq p$. It follows that for all $\omega' \in Ben_i(\omega)$ we have $t_i(\omega')(E) \geq p$. Hence $Ben_i(\omega) \subseteq B_i^p(E)$. Thus $t_i(\omega)(B_i^p(E)) = 1$, which implies that $\omega \in B_i^1 B_i^p(E)$. $\square$

C.5 Proof of Proposition 8

1. This is property is equivalent to $B_i^p(E) \cup B_i^p \neg B_i^p(E) \subseteq A_i(E)$. By Property 5. we have $B_i^p(E) \subseteq A_i(E)$. To see that $B_i^p \neg B_i^p(E) \subseteq A_i(E)$, note that $\omega \in B_i^p \neg B_i^p(E)$ if and only if $t_i(\omega)(\neg B_i^p(E)) \geq p$. This implies that $S_{t_i(\omega)} \succeq S(\neg B_i^p(E)) = S(E)$. The last equality follows by Property 8 and Proposition 2. Hence $\omega \in A_i(E)$.

2. The proof is analogous to 1. The is property is equivalent to $\bigcap_{n=1}^{\infty} B_i^p \left(\neg B_i^p\right)^{n-1}(E) \subseteq A_i(E)$. $\omega \in B_i^p \left(\neg B_i^p\right)^{n-1}(E)$ for any $n = 1, 2, ...$ if and only it $t_i(\omega)\left(\left(\neg B_i^p\right)^{n-1}(E)\right) \geq p$ for any $n = 1, 2, ...$ It follows that $S_{t_i(\omega)} \succeq S\left(\left(\neg B_i^p\right)^{n-1}(E)\right)$ for any $n = 1, 2, ...$ By Proposition 2, $S\left(\left(\neg B_i^p\right)^{n-1}(E)\right) = S(E)$ for any $n = 1, 2, ...$ Hence $\omega \in A_i(E)$.

3. First, we show $B_i^p U_i(E) \subseteq A_i(E)$. $\omega \in B_i^p U_i(E)$ if and only if $t_i(\omega)(U_i(E)) \geq p$. It implies $S_{t_i(\omega)} \succeq S(U_i(E))$. By Proposition 1 $S(U_i(E)) = S(E)$. Hence $S_{t_i(\omega)} \succeq S(E)$ which is equivalent to $\omega \in A_i(E)$.

Second, we show that $B_i^p U_i(E) = \emptyset^{S(E)}$ for $p \in (0, 1]$. Since $B_i^p U_i(E) \subseteq A_i(E)$ we have by monotonicity $B_i^1 B_i^p U_i(E) \subseteq B_i^1 A_i(E)$. By introspection $B_i^p U_i(E) \subseteq B_i^1 B_i^p U_i(E) \subseteq B_i^1 A_i(E)$. By additivity, we have $B_i^p U_i(E) \subseteq \neg B_i^1 A_i(E)$. Hence $B_i^p U_i(E) = \emptyset^{S(E)} = \neg B_i^1 A_i(E) \cap B_i^1 A_i(E)$.

Third, we show that $B_i^0 U_i(E) = A_i(E)$. $\omega \in A_i(E)$ if and only if $\omega \in A_i U_i(E)$ since by AA-self-reflection $A_i(E) = A_i A_i(E)$ and by symmetry $A_i A_i(E) = A_i U_i(E)$. Hence, if $\omega \in A_i(E)$ then $t_i(\omega)(U_i(E))$ is defined. Therefore $\omega \in B_i^0 U_8 E)$, and hence $A_i(E) \subseteq B_i^0 U_i(E)$. Together with the first part of the proof, we conclude $B_i^0 U_i(E) = A_i(E)$.

4. This is property is equivalent to $A_i U_i(E) = A_i(E)$. $\omega \in A_i U_i(E)$ if and only if $S_{t_i(\omega)} \succeq S(U_i(E)) = S(A_i(E)) = S(E)$ by Proposition 1. Hence $\omega \in A_i U_i(E)$ if and only if $\omega \in A_i(E)$.

5. $\omega \in A_i(E)$ if and only if $S_{t_i(\omega)} \succeq S(E)$. For any $t_i(\omega)$, we have $S_{t_i(\omega)} \succeq S(E)$ if and only if $1 = t_i(\omega)(S(E)^{\uparrow})$. This is equivalent to $\omega \in B_i^1(S(E)^{\uparrow})$.

6. First, we show $B_i^p(E) \subseteq A_i(E)$. $\omega \in B_i^p(E)$ if and only if $t_i(\omega)(E) \geq p$. This implies that $S_{t_i(\omega)} \succeq S(E)$, which is equivalent to $\omega \in A_i(E)$.

Second, we show for $p = 0$, $A_i(E) \subseteq B_i^0(E)$. $\omega \in A_i(E)$ if and only if $t_i(\omega) \in \Delta(S)$ with $S \succeq S(E)$. Hence $t_i(\omega)(E) \geq 0$, which implies that $\omega \in B_i^0(E)$.

7. $\omega \in B_i^p(E)$ if and only if $t_i(\omega)(E) \geq p$. This implies that $S_{t_i(\omega)} \succeq S(E)$. By Proposition 2 it is equivalent to $S_{t_i(\omega)} \succeq S(B_i^q(E))$, which is equivalent to $\omega \in A_i B_i^q(E)$.

8. By the definition of negation, $S(E) = S(\neg E)$. Hence for $t_i(\omega) \in \triangle(S)$, $S \succeq S(E)$ if and only if $S \succeq S(\neg E)$.

9. $\omega \in \bigcap_{\lambda \in L} A_i(E_\lambda)$ if and only if $S_{t_i(\omega)} \succeq S(E_\lambda)$ for all $\lambda \in L$. This is equivalent to $S_{t_i(\omega)} \succeq \sup_{\lambda \in L} S(E_\lambda) = S\left(\bigcap_{\lambda \in L} E_\lambda\right)$, which is equivalent to $\omega \in A_i\left(\bigcap_{\lambda \in L} E_\lambda\right)$.

10. By Proposition 2, $S(E) = S(B_i^p(E))$. Hence, $\omega \in A_i(E)$ if and only if $\omega \in A_i B_i^p(E)$.

11. By Proposition 1, $S(E) = S(A_i(E))$. Hence $\omega \in A_i(E)$ if and only if $\omega \in A_i A_i(E)$.

12. $\omega \in B_i^p A_i(E)$ if and only if $t_i(\omega)(A_i(E)) \geq p$. This implies $S_{t_i(\omega)} \succeq S(A_i(E))$. By Proposition 1, $S(A_i(E)) = S(E)$. Thus $\omega \in A_i(E)$. To see the converse, by weak necessitation and introspection, $A_i(E) = B_i^1(S(E)^{\uparrow}) \subseteq B_i^1 B_i^1(S(E)^{\uparrow}) = B_i^1 A_i(E)$. By Proposition 7 (o), $B_i^1 A_i(E) \subseteq B_i^p A_i(E)$. $\square$

C.6 Proof of Proposition 10

1. By Proposition 1, $S(E) = S(A_j(E))$. Hence $\omega \in A_i(E)$ if and only if $\omega \in A_i A_j(E)$.

2. By Proposition 2, $S(E) = S(B_j^p(E))$. Hence, $\omega \in A_i(E)$ if and only if $\omega \in A_i B_j^p(E)$.

3. $\omega \in B_i^p(E)$ if and only if $t_i(\omega)(E) \geq p$. This implies that $S_{t_i(\omega)} \succeq S(E)$. By Proposition 2, this

is equivalent to $S_{t_i(\omega)} \succeq S(B_j^q(E))$, which is equivalent to $\omega \in A_i B_j^q(E)$.

4. The proof is analogous to 3.

5. We show by induction that $A^n(E) = A(E)$, for all $n \geq 1$. We have $\omega \in A(A^n(E))$ if and only if $S_{t_i(\omega)} \succeq S(A^n(E))$, for all $i \in I$, which, by the induction hypothesis, is the case if and only if $S_{t_i(\omega)} \succeq S(A(E))$, for all $i \in I$. By the definition of "$\cap$", it is the case that $S(A(E)) = \sup_{i \in I} S(A_i(E))$. By Proposition 1 we have $S(A_i(E)) = S(E)$ and hence $S(A(E)) = S(E)$. It follows that $S_{t_i(\omega)} \succeq S(A(E))$ if and only if $S_{t_i(\omega)} \succeq S(E)$. But $S_{t_i(\omega)} \succeq S(E)$ if and only if $\omega \in A_i(E)$. Hence we have $A^n(E) = A(E)$, for all $n \geq 1$, and therefore $CA(E) = A(E)$.

6. $\omega \in CB^1(E)$ implies $\omega \in B_i^1(E)$ for all $i \in I$. This is equivalent to $t_i(\omega)(E) = 1$ for all $i \in I$, which implies $S_{t_i(\omega)} \succeq S(E)$ for all $i \in I$. Hence, by 5. we have $\omega \in A(E) = CA(E)$.

7. First, we show that $B^p(E) \subseteq A(E)$. $\omega \in B^p(E)$ if and only if $t_i(\omega)(E) \geq p$ for all $i \in I$. Hence $t_i(\omega) \in \Delta(S)$ with $S \succeq S(E)$, for all $i \in I$. This implies that $\omega \in A_i(E)$, for all $i \in I$. It follows that $\omega \in A(E)$.

Second, we show that $A(E) = B^0(E)$. $\omega \in A(E)$ if and only if $\omega \in A_i(E)$ for all $i \in I$ if and only if (by 6. of Proposition 8) $\omega \in B_i^0(E)$ for all $i \in I$ if and only if $\omega \in B^0(E)$.

8. The proof follows from 7. and 5.

9. By weak necessitation, $A(E) := \bigcap_{i \in I} A_i(E) = \bigcap_{i \in I} B_i^1(S(E)^{\uparrow}) := B^1(S(E)^{\uparrow})$.

10. The proof follows from 9. and 5.

11. By definition of common certainty, $CB^1(S(E)^{\uparrow}) \subseteq B^1(S(E)^{\uparrow})$. By 9., $B^1(S(E)^{\uparrow}) = A(E)$.

12. The proof follows from 11. and 5. $\square$

C.7 Proof of Proposition 4

We state the proof for Bayesian games with unawareness (allowing also for unawareness of actions and players).

Let $\sigma^*_{|\Omega'}$ be an equilibrium in the S'-partial Bayesian game with unawareness $\Gamma(\underline{\Omega}')$. For $S'' \succeq S'$ we define a strategic form game with

- $I(\Omega'' \setminus \Omega') := \{(i, t_i(\omega)) : \omega \in \Omega'', i \in I(\omega)\} \setminus \{(i, t_i(\omega)) : \omega \in \Omega', i \in I(\omega)\}$ being the set of players,
- the set of strategies of player $(i, t_i(\omega)) \in I(\Omega'', \Omega')$ is $\Delta(\mathcal{M}_i(\omega))$,
- the payoff function of player $(i, t_i(\omega))$ is given by equation (1) but fixing the strategy of each (dummy) player in $\{(i, t_i(\omega')) : \omega' \in \Omega', i \in I(\omega')\}$ to her respective equilibrium strategy $\sigma_i^*(\omega)$ of the S'-partial Bayesian game with unawareness $\Gamma(\underline{\Omega}')$.

Since I, Ω, and $(M_i)_{i \in I}$ are finite, this strategic game has an equilibrium by Nash's (1950) theorem.

Consider now the strategy profile $\sigma^*_{|\Omega''}$ in which players in $\{(i, t_i(\omega)) : \omega \in \Omega', i \in I(\omega)\}$ play their component of the profile $\sigma^*_{|\Omega'}$ and players in $I(\Omega'' \setminus \Omega')$ play the equilibrium strategies of the equilibrium in above defined strategic game.

We need to show that $\sigma^*_{|\Omega''}$ is an equilibrium of the S''-partial Bayesian game with unawareness $\Gamma(\underline{\Omega}'')$. Suppose not, then for some player $(i, t_i(\omega)) \in I(\Omega'') = \{(i, t_i(\omega')) : \omega' \in \Omega'', i \in I(\omega')\}$ there exists $\sigma_i(\omega) \in \Delta(\mathcal{M}_i(\omega))$ with $\sigma_i(\omega) \neq \sigma_i^*(\omega)$ such that for $\sigma_{S_{t_i(\omega)}} := \left(\sigma_i(\omega), (\sigma_j^*(\omega'))_{\omega' \in S_{t_i(\omega)}, j \in I(\omega) \setminus \{i\}}\right)$ we have

$$U_{(i, t_i(\omega))}(\sigma_{S_{t_i(\omega)}}) > U_{(i, t_i(\omega))}(\sigma^*_{S_{t_i(\omega)}}),$$

i.e., there exists a profitable deviation from $\sigma^*_{|\Omega''}$ for some player-type $(i, t_i(\omega))$ with $\omega \in \Omega''$ and $i \in I(\omega)$ given that all other player-types in $I(\Omega'')$ play their equilibrium strategy.

If $(i, t_i(\omega)) \in I(\Omega'' \setminus \Omega')$ then her strategy is not an equilibrium strategy in above defined strategic game, a contradiction. If $(i, t_i(\omega)) \in \{(i, t_i(\omega)) : \omega' \in \Omega', i \in I(\omega')\}$, then since her payoffs are identical in both games, her strategy is not an equilibrium strategy in the S'-partial Bayesian game with unawareness $\Gamma(\underline{\Omega}')$, a contradiction. Hence $\sigma^*_{|\Omega''}$ must be an equilibrium of the S''-partial Bayesian game with unawareness $\Gamma(\underline{\Omega}'')$. □

C.8 Proof of Proposition 5

We state the proof for Bayesian games with unawareness (allowing also for unawareness of actions and players).

Let $\sigma^*_{|\Omega''}$ be an equilibrium of the S''-partial Bayesian game with unawareness $\Gamma(\underline{\Omega}'')$. Moreover, let $\sigma^*_{|\Omega'}$ be a profile of strategies that is identical with $\sigma^*_{|\Omega''}$ for all $(i, t_i(\omega)) \in I(\Omega')$.

Suppose to the contrary that $\sigma^*_{|\Omega'}$ is not an equilibrium of the S'-partial Bayesian game with unawareness $\Gamma(\underline{\Omega}')$. Then for some player $(i, t_i(\omega)) \in I(\Omega')$ there exists $\sigma_i(\omega) \in \Delta(\mathcal{M}_i(\omega))$ with $\sigma_i(\omega) \neq \sigma^*_i(\omega)$ such that for $\sigma_{S_{t_i(\omega)}} := \left(\sigma_i(\omega), (\sigma^*_j(\omega'))_{\omega' \in S_{t_i(\omega)}, j \in I(\omega) \setminus \{i\}}\right)$ we have

$$U_{(i,t_i(\omega))}(\sigma_{S_{t_i(\omega)}}) > U_{(i,t_i(\omega))}(\sigma^*_{S_{t_i(\omega)}}),$$

i.e., there exists a profitable deviation from $\sigma^*_{|\Omega'}$ for some player-type $(i, t_i(\omega))$ with $\omega \in \Omega'$ and $i \in I(\omega)$. This is a contradiction to $\sigma^*_i(\omega)$ being an equilibrium strategy in the S''-partial Bayesian game with unawareness $\Gamma(\Omega'')$, because fixing the strategies of the other players, the payoffs to this player are the same in both games. □

C.9 Proof of Proposition 6

Before we prove the proposition, we require following auxiliary results:

Remark 2 *For any $\omega \in \Omega$, $t_i(\omega)(E \cap A_i(E)) = t_i(\omega)(E)$ for any event E s.t. $S(E) \preceq S_{t_i(\omega)}$.*

Proof of the Remark: Let E be an event and $t_i(\omega)$ be such that $S(E) \preceq S_{t_i(\omega)}$. Since $E = (E \cap A_i(E)) \cup (E \cap U_i(E))$ and $A_i(E) \cap U_i(E) = \emptyset^{S(E)}$, we have $(E \cap A_i(E)) \cap (E \cap U_i(E)) = \emptyset^{S(E)}$. Since $t_i(\omega)$ is an additive probability measure, $t_i(\omega)(E) = t_i(\omega)(E \cap A_i(E)) + t_i(\omega)(E \cap U_i(E))$. Since $B^p_i U_i(E) = \emptyset^{S(E)}$ for $p \in (0, 1]$ ($B^p U$-Introspection in Proposition 8), we must have $t_i(\omega)(E \cap U_i(E)) = 0$. □

We slightly abuse terminology and call a probability measure $\mu_i \in \Delta(S)$ a *prior for player i on S* if for every event $E \in \Sigma$ with $S(E) \preceq S$ equation (2u) is satisfied, i.e.,

$$\mu_i(E \cap S \cap A_i(E)) = \int_{S \cap A_i(E)} t_i(\cdot)(E) d\mu_i(\cdot). \tag{5}$$

The following lemma says that if there is a prior on a state-space then the marginal on a lower space is a prior as well.

Lemma 2 *If $\mu_i \in \Delta(S')$ is a prior for player i on S' and $S \preceq S'$, then $(\mu_i)_{|S}$ (the marginal of μ_i on S) is a prior for player i on S.*

Proof of the Lemma. Let E be an event with $S(E) \preceq S$ and let μ be individual i's probability measure on S' with $S' \succeq S$. Since $(\mu)_{|S}$ is a probability measure on S, $(\mu)_{|S}(A_i(E) \cap E) = (\mu)_{|S}(A_i(E) \cap$

$E\cap S$) (we do not like to abuse notation for what follows). By the definition of the marginal, $(\mu)_{|S}(A_i(E)\cap E\cap S) = \mu\left((r_S^{S'})^{-1}(A_i(E)\cap E\cap S)\right)$. Since μ is a probability measure on S' satisfying equation (5), the fact that $A_i(E)$ is an event, and Remark 2,

$$\mu\left((r_S^{S'})^{-1}(A_i(E)\cap E\cap S)\right) = \int_{\omega'\in A_i(E)\cap S'} t_i(\omega')\left((r_S^{S'})^{-1}(A_i(E)\cap E\cap S)\right) d\mu(\omega').$$

By Property (1) of t_i,

$$\begin{aligned} &\int_{\omega'\in A_i(E)\cap S'} t_i(\omega')\left((r_S^{S'})^{-1}(A_i(E)\cap E\cap S)\right) d\mu(\omega') \\ &= \int_{\omega'\in A_i(E)\cap S'} t_i(r_S^{S'}(\omega'))(A_i(E)\cap E\cap S)d\mu(\omega'). \end{aligned}$$

By the definition of marginal,

$$\begin{aligned} &\int_{\omega'\in A_i(E)\cap S'} t_i(r_S^{S'}(\omega'))(A_i(E)\cap E\cap S)d\mu(\omega') \\ &= \int_{\omega\in A_i(E)\cap S} t_i(\omega)(A_i(E)\cap E\cap S)d(\mu)_{|S}(\omega) \\ &= (\mu)_{|S}(A_i(E)\cap E\cap S) = (\mu)_{|S}(A_i(E)\cap E). \end{aligned}$$

□

We say that $\mu\in\Delta(S)$ is a common prior on S if it is a prior on S for every player $i\in I$.

Remark 3 *Let $\hat{S}$ be the upmost state space in the lattice $\mathcal{S}$, and let $(P_i^S)_{S\in\mathcal{S}}\in\prod_{S\in\mathcal{S}}\Delta(S)$ be a tuple of probability measures. Then $(P_i^S)_{S\in\mathcal{S}}$ is a prior for player i if and only if $P_i^{\hat{S}}$ is a prior for player i on $\hat{S}$ and P_i^S is the marginal of $P_i^{\hat{S}}$ for every $S\in\mathcal{S}$.*

This remark together with Lemma 2 implies the following:

Remark 4 *A common prior (Definition 10) induces a common prior on S, for any $S\in\mathcal{S}$. The converse is not necessarily true unless S is the upmost state-space of the lattice. Note that it is possible that players have different priors, but at some space S (below the upmost space) the priors on S coincide. Hence, in such a case they have different priors, but a common prior on S (and by Lemma 2 also a common prior on spaces less expressive than S).*

We are now ready to prove Proposition 6. In fact, we prove below a version just requiring the existence of a common prior P^S on S such that $S(G)\preceq S$ and $P^S(CB^1(\bigcap_{i\in I}[t_i(G)=p_i]))>0$. By Remark 4, this is more general than the statement of Proposition 6.

Proof of Proposition 6. By Proposition 9, $\omega\in CB(F)$ if and only if there exists an event E that is evident such that $\omega\in E\subseteq B^1(F)$.

Since for an evident E we have $E\subseteq B_i^1(E)\subseteq A_i(E)$ for all $i\in I$. It follows that $P^S(E\cap A_i(E)) = P^S(E)$ for $S\succeq S(E)$. Set $F=\bigcap_{i\in I}[t_i(G)=p_i]$ and let $E=CB(F)$. By Proposition 1, $S(E)=S(G)$. By Lemma 2 and the properties imposed on t_i, we consider w.l.o.g. a common prior $P^{S(G)}$ on $S(G)$.

$$P^{S(G)}(E) = \int_{S(G)\cap A_i(E)} t_i(\cdot)(E)dP^{S(G)}(\cdot)$$

$$= \int_{E\cap S(G)\cap A_i(E)} t_i(\cdot)(E)dP^{S(G)}(\cdot) + \int_{(S(G)\cap A_i(E))\setminus E} t_i(\cdot)(E)dP^{S(G)}(\cdot).$$

$$\int_{E\cap S(G)\cap A_i(E)} t_i(\cdot)(E)dP^{S(G)}(\cdot) = \int_{E\cap S(G)\cap A_i(E)} 1dP^{S(G)}(\cdot) = P^{S(G)}(E).$$

It follows that

$$\int_{(S(G)\cap A_i(E))\setminus E} t_i(\cdot)(E)dP^{S(G)}(\cdot) = 0. \tag{6}$$

$$\int_{E\cap A_i(E)\cap S(G)} t_i(\cdot)(G)dP^{S(G)}(\cdot) = \int_{E\cap A_i(E)\cap S(G)} p_i dP^{S(G)}(\cdot) = p_i P^{S(G)}(E)$$

$$\begin{aligned}\int_{E\cap A_i(E)\cap S(G)} t_i(\cdot)(G)dP^{S(G)}(\cdot) &= \int_{E\cap A_i(E)\cap S(G)} t_i(\cdot)(G\cap E)dP^{S(G)}(\cdot)\\ &= \int_{S(G)\cap A_i(E)} t_i(\cdot)(G\cap E)dP^{S(G)}(\cdot) - \int_{(S(G)\cap A_i(E))\setminus E} t_i(\cdot)(G\cap E)dP^{S(G)}(\cdot)\end{aligned}$$

Since by the monotonicity of probability measures

$$\int_{(S(G)\cap A_i(E))\setminus E} t_i(\cdot)(G\cap E)dP^{S(G)}(\cdot) \leq \int_{(S(G)\cap A_i(E))\setminus E} t_i(\cdot)(E)dP^{S(G)}(\cdot),$$

we must have by equation (6) and non-negativity of probability measures

$$\int_{(S(G)\cap A_i(E))\setminus E} t_i(\cdot)(G\cap E)dP^{S(G)}(\cdot) = 0.$$

Note that $P^{S(G)}(G\cap E) = \int_{S(G)\cap A_i(E)} t_i(\cdot)(G\cap E)dP^{S(G)}(\cdot)$.

Note further that $P^{S(G)}(E) = P^{S(G)}(E\cap A_i(E))$ for all $i \in N$ since $E = CB^1(F) \subseteq A_i(E)$ for all $i \in N$. Similarly, $P^{S(G)}(G\cap E) = P^{S(G)}(G\cap E\cap A_i(E))$ for all $i \in N$.

Thus

$$p_i P^{S(G)}(E) = P^{S(G)}(G\cap E). \tag{7}$$

Note that by assumption $P^{S(G)}(E) > 0$.

Since equation (7) holds for all $i \in I$, we must have $p_i = p_j$, for all $i, j \in I$. □

C.10 Proof of Theorem 1

Before we prove the theorem, we state following observations:

Remark 5 *If $P = \left(P^S\right)_{S\in\mathcal{S}} \in \prod_{S\in\mathcal{S}} \Delta(S)$ is a non-degenerate (common) prior, then also $P^S \in \Delta(S)$ is non-degenerate (common) prior on S for every $S \in \mathcal{S}$.*

Remark 6 *If $\mu_i \in \Delta(S)$ is a* non-degenerate *prior for player i on S and $S' \preceq S$, then the marginal of μ_i on S', $\left(\mu_i^S\right)_{|S'}$ is a* non-degenerate *prior for player i on S'.*

Lemma 3 *Let P^S be a non-degenerate common prior on some finite state-space S and let $i \in I$ and $\omega \in \Sigma$ such that $t_i(\omega) \in \triangle(S)$. Then we have for all $\omega' \in [t_i(\omega)] \cap S$ that $t_i(\omega)(\{\omega'\}) = \frac{P^S(\{\omega'\})}{P^S([t_i(\omega)]\cap S)}$.*

Proof of the Lemma. Because $t_i(\omega) = t_i(\omega')$, we have $A_i(S^{\uparrow}) = A_i\left(\{\omega'\}^{\uparrow}\right) \supseteq [t_i(\omega)]^{\uparrow} \supseteq \{\omega'\}^{\uparrow}$. By the definition of a prior on S, $P^S(\{\omega'\}) = P^S\left(\{\omega'\}^{\uparrow} \cap A_i\left(\{\omega'\}^{\uparrow}\right)\right) = \int\limits_{A_i\left(\{\omega'\}^{\uparrow}\right)\cap S} t_i(\cdot)\left(\{\omega'\}^{\uparrow}\right) dP^S(\cdot)$. Note that if $\omega'' \in S \setminus [t_i(\omega)] \cap S$, then we do have $t_i(\omega'')\left(\{\omega'\}^{\uparrow}\right) = 0$. Hence, since $t_i(\omega) = t_i(\omega'')$, for $\omega'' \in [t_i(\omega)]$, we have $\int\limits_{A_i\left(\{\omega'\}^{\uparrow}\right)\cap S} t_i(\cdot)\left(\{\omega'\}^{\uparrow}\right) dP^S(\cdot) = \int\limits_{[t_i(\omega)]\cap S} t_i(\cdot)(\{\omega'\})\, dP^S(\cdot) = t_i(\omega)(\{\omega'\})\, P^S([t_i(\omega)] \cap S)$. Because P^S is non-degenerate, it follows that $t_i(\omega)(\{\omega'\}) = \frac{P^S(\{\omega'\})}{P^S([t_i(\omega)]\cap S)}$.
□

Proof of the Theorem. Note that $E_1^{>\alpha}$ and $E_2^{\leq\alpha}$ may not be events in our unawareness belief structure. In Meier and Schipper (2007) we extend the definition of the belief operator as well as Proposition 7 and 9 to measurable subsets of Ω.

Suppose that $CB\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right)$ is non-empty. Then fix a $\preceq$-minimal state-space S such that $W := CB\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right) \cap S \neq \emptyset$. Such a space S exists by the finiteness of Σ.

By Remark 5, since P is non-degenerate common prior, P^S is a non-degenerate common prior on S.

Since $W = CB\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right) \cap S \subseteq S \cap B_i^1\left(CB\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right)\right)$, the minimality of S implies that for each $\omega \in CB\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right) \cap S$ we do have $S_{t_i(\omega)} = S$ and $t_i(\omega)(W) = 1$.

By the definition, $t_i(\omega)([t_i(\omega)] \cap S) = 1$, for each $\omega \in CB\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right) \cap S$. Since $t_i(\omega)(W) = 1$, we have $t_i(\omega)(([t_i(\omega)] \cap S) \setminus W) = 0$.

By Lemma 3, this implies that $P^S(\{\omega'\}) = 0$, for $\omega' \in ([t_i(\omega)] \cap S) \setminus W$ such that $\omega \in CB\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right) \cap S$. It follows that $P^S(([t_i(\omega)] \cap S) \setminus W) = 0$ and hence, $P^S(([t_i(\omega)] \cap S) \cap W) = P^S(([t_i(\omega)] \cap S) \cap W) + P^S(([t_i(\omega)] \cap S) \setminus W) = P^S(([t_i(\omega)] \cap S)) > 0$. So, we do have $P^S(W) > 0$.

The fact that $P^S(\{\omega'\}) = 0$, for $\omega' \in ([t_i(\omega)] \cap S) \setminus W$ such that $\omega \in CB\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right) \cap S = W$ implies the following: For any random variable x, we have $\sum_{\omega' \in [t_i(\overline{\omega})] \cap S} x(\omega') P^S(\{\omega'\}) = \sum_{\omega' \in W \cap [t_i(\overline{\omega})] \cap S} x(\omega') P^S(\{\omega'\})$, if $[t_i(\overline{\omega})] \cap W \neq \emptyset$. And also $\sum_{\omega \in W} x(\omega) P^S(\{\omega\}) = \sum_{[t_i(\overline{\omega})] \cap W \neq \emptyset} \sum_{\omega \in [t_i(\overline{\omega})] \cap S} x(\omega) P^S(\{\omega\})$. This is so, because there is a $\omega \in [t_i(\overline{\omega})] \cap W$ and for this ω, we do have $\omega \in CB\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right) \cap S$ and $[t_i(\omega)] = [t_i(\overline{\omega})]$ and this implies $P^S(([t_i(\overline{\omega})] \cap S) \setminus W) = 0$.

For $i = 1, 2$ we have

$$\begin{aligned}
&\sum_{\omega \in W} P^S(\{\omega\}) \sum_{\omega' \in [t_i(\omega)] \cap S} v(\omega')\, t_i(\omega)(\{\omega'\}) \\
&= \sum_{\omega \in W} P^S(\{\omega\}) \sum_{\omega' \in [t_i(\omega)] \cap S} v(\omega') \frac{P^S(\{\omega'\})}{P^S([t_i(\omega)] \cap S)} \\
&= \sum_{[t_i(\overline{\omega})] \cap W \neq \emptyset} \sum_{\omega \in [t_i(\overline{\omega})] \cap S} P^S(\{\omega\}) \sum_{\omega' \in [t_i(\omega)] \cap S} v(\omega') \frac{P^S(\{\omega'\})}{P^S([t_i(\omega)] \cap S)} \\
&= \sum_{[t_i(\overline{\omega})] \cap W \neq \emptyset} \sum_{\omega \in [t_i(\overline{\omega})] \cap S} P^S(\{\omega\}) \sum_{\omega' \in [t_i(\overline{\omega})] \cap S} v(\omega') \frac{P^S(\{\omega'\})}{P^S([t_i(\omega)] \cap S)} \\
&= \sum_{[t_i(\overline{\omega})] \cap W \neq \emptyset} P^S([t_i(\overline{\omega})] \cap S) \sum_{\omega' \in [t_i(\overline{\omega})] \cap S} v(\omega') \frac{P^S(\{\omega'\})}{P^S([t_i(\overline{\omega})] \cap S)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{[t_i(\varpi)]\cap W\neq\emptyset} \sum_{\omega'\in[t_i(\varpi)]\cap S} v(\omega') P^S(\{\omega'\}) \\
&= \sum_{\omega'\in W} v(\omega') P^S(\{\omega'\}).
\end{aligned}
$$

But by the assumptions, we have $\sum_{\omega\in W} P^S(\{\omega\}) \sum_{\omega'\in[t_1(\omega)]\cap S} v(\omega') t_1(\omega)(\{\omega'\}) > \alpha P^S(W)$ and $\sum_{\omega\in W} P^S(\{\omega\}) \sum_{\omega'\in[t_2(\omega)]\cap S} v(\omega') t_2(\omega)(\{\omega'\}) \leq \alpha P^S(W)$, a contradiction, since $P^S(W) > 0$. $\square$

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Appendix to Unawareness, Beliefs and Games: Speculative Trade under Unawareness - The Infinite Case*

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We generalize the "No-trade" theorem for finite unawareness belief structures in Heifetz, Meier and Schipper (2007) to the infinite case. We also discuss a technical device called the "flattened type-space" associated with an unawareness belief structure, in which the state-space is the union of spaces in the lattice.

1 Topological Unawareness Belief Structures

We consider an unawareness belief structure as defined in Heifetz, Meier and Schipper (2007). In addition we impose the following conditions:

1. The set of individuals I is at most countable.
2. Each state-space $S \in \mathcal{S}$ is a non-empty compact Hausdorff space.
3. $(\mathcal{S}, \preceq)$ is well-founded, that is, every non-empty subset $\mathcal{X} \subseteq \mathcal{S}$ contains a $\preceq$-minimal element. (That is, there is a $S' \in \mathcal{X}$ such that for all $S \in \mathcal{X}$: if $S \preceq S'$, then $S = S'$.)
4. For all $S, S' \in \mathcal{S}$ with $S' \preceq S$, we have that $r^S_{S'}$ is continuous and surjective.
5. For each $S \in \mathcal{S}$, $\Delta(S)$ is the space of *regular* Borel probability measures on S, which is endowed with the topology of weak convergence.[1]

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[1] This topology is generated by the sub-basis of sets of the form

$$\{\mu \in \Delta(S) : \mu(O) > r\}$$

where $O \subseteq S$ is open and $r \in \mathbb{R}$ (see e.g. Billingsley (1968), appendix III). When S is Normal (and in particular

6. For each player $i \in I$, $t_i : \Omega \longrightarrow \bigcup_{S \in \mathcal{S}} \Delta(S)$ is continuous, where

   a. Ω is endowed with the disjoint-union topology: $O \subseteq \Omega$ is open if and only if $O \cap S$ is open in S for all $S \in \mathcal{S}$.

   b. $\bigcup_{S \in \mathcal{S}} \Delta(S)$ is endowed with the disjoint-union topology: $O_\Delta \subseteq \bigcup_{S \in \mathcal{S}} \Delta(S)$ is open if and only if $O_\Delta \cap \Delta(S)$ is open in $\Delta(S)$ for all $S \in \mathcal{S}$.

We call an unawareness belief structure satisfying above conditions a *topological unawareness belief structure.*

Note that although each S and each $\Delta(S)$ are compact, if $\mathcal{S}$ is infinite, Ω and $\bigcup_{S \in \mathcal{S}} \Delta(S)$ are *not* compact.

2 A Generalized "No-Trade" Theorem

Definition 1 (Prior) *A prior for player i is a system of probability measures $P_i = \left(P_i^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ such that*

1. *The system is projective: If $S' \preceq S$ then the marginal of P_i^S on S' is $P_i^{S'}$. (That is, if $E \in \Sigma$ is an event whose base-space $S(E)$ is lower or equal to S', then $P_i^S(E) = P_i^{S'}(E)$.)*

2. *Each probability measure P_i^S is a convex combination of i's beliefs in S: For every event $E \in \Sigma$ such that $S(E) \preceq S$,*

$$P_i^S\left(E \cap S \cap A_i(E)\right) = \int_{S \cap A_i(E)} t_i(\cdot)(E)\, dP_i^S(\cdot). \tag{1}$$

We call any probability measure $\mu_i \in \Delta(S)$ satisfying equation (1) in place of P_i^S a prior of player i on S.

Definition 2 (Common Prior) *$P = \left(P^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ (resp. $P^S \in \Delta(S)$) is a common prior (resp. a common prior on S) if P (resp. P^S) is a prior for every player $i \in I$.*

Definition 3 *A common prior $P = \left(P^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ (resp. a common prior P^S on S) is* non-degenerate *if and only if for all $i \in I$ and $\omega \in \Omega$: If $t_i(\omega) \in \triangle(S')$, for some S', then $P^S\left(\left([t_i(\omega)] \cap S'\right)^{\uparrow} \cap S\right) > 0$ for all $S \succeq S'$.*

Note that by Lemma 3 below, $[t_i(\omega)] \cap S' \in \mathcal{F}_{S'}$.

---

compact and/or metric), this topology coincides with the weak* topology - the weakest topology for which the mapping

$$\mu \longrightarrow \int_S f d\mu$$

is continuous for every continuous real-valued function f on S.

Recall Remark 3 in Heifetz, Meier and Schipper (2007) according to which if $\hat{S}$ is the upmost state-space in the lattice $\mathcal{S}$, and $(P_i^S)_{S\in\mathcal{S}} \in \prod_{S\in\mathcal{S}} \Delta(S)$ is a tuple of probability measures, then $(P_i^S)_{S\in\mathcal{S}}$ is a prior for player i if and only if $P_i^{\hat{S}}$ is a prior for player i on $\hat{S}$ and P_i^S is the marginal of $P_i^{\hat{S}}$ for every $S \in \mathcal{S}$.

Definition 4 *Let x_1 and x_2 be real numbers and v a continuous random variable on Ω. Define the sets* $E_1^{\leq x_1} := \left\{\omega \in \Omega : \int_{S_{t_1(\omega)}} v(\cdot)\, d(t_1(\omega))(\cdot) \leq x_1\right\}$ *and* $E_2^{\geq x_2} := \left\{\omega \in \Omega : \int_{S_{t_2(\omega)}} v(\cdot)\, d(t_2(\omega))(\cdot) \geq x_2\right\}$. *We say that at ω, conditional on his information, player 1 (resp. player 2) believes that the expectation of v is weakly below x_1 (resp. weakly above x_2) if and only if $\omega \in E_1^{\leq x_1}$ (resp. $\omega \in E_1^{\geq x_2}$).*

Theorem 1 *Let $\underline{\Omega}$ be a topological unawareness belief structure and P a non-degenerate common prior. Then there is no state $\tilde{\omega} \in \Omega$ such that there are a continuous random variable $v : \Omega \longrightarrow \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$, with the following property: at $\tilde{\omega}$ it is common certainty that conditional on her information, player 1 believes that the expectation of v is weakly below x_1 and, conditional on his information, player 2 believes that the expectation of v is weakly above x_2.*

This general "No-trade" theorem implies our "No-trade" theorem for finite unawareness belief structures (Heifetz, Meier and Schipper, 2007).

Before we prove Theorem 1, we introduce some technical devices.

3 The Flattened Structure

Definition 5 *$G \subseteq \Omega$ is a* measet *if and only if for all $S \in \mathcal{S}$, $G \cap S \in \mathcal{F}_S$.*

Remark 1 *The collection of measets forms a sigma-algebra on Ω.*

Remark 2 *Let $\mathcal{S}$ be at most countable and G be a measet, $p \in [0,1]$ and $i \in I$. Then $\{\omega \in \Omega : t_i(\omega)(G) \geq p\}$ is a measet.*

Let $\underline{\Omega}$ be an unawareness belief structure. We the define the *flattened type-space* associated with the unawareness belief structure $\underline{\Omega}$ by

$$F(\underline{\Omega}) := \langle \Omega, \mathcal{F}, (t_i^F)_{i\in I} \rangle,$$

where

- Ω is the union of all state-spaces in $\underline{\Omega}$,
- $\mathcal{F}$ is the collection of all measets in $\underline{\Omega}$, and
- $t_i^F : \Omega \longrightarrow \Delta(\Omega, \mathcal{F})$ is defined as follows: $t_i^F(\omega)(E) := t_i(\omega)(E \cap S_{t_i(\omega)})$, if $E \cap S_{t_i(\omega)} \neq \emptyset$, and zero otherwise.

A *standard type-space* on S for the player set I is a tuple

$$\underline{Y} := \left\langle Y, \mathcal{F}_Y, (t_i)_{i\in I} \right\rangle,$$

where

- Y is a nonempty set,
- $\mathcal{F}_Y$ is a sigma-field on Y,
- for $i \in I : t_i$ is a $\mathcal{F}_Y - \mathcal{F}_{\Delta(Y)}$ -measurable function from Y to $\Delta(Y, \mathcal{F}_Y)$, the space of countable additive probability measures on $(Y, \mathcal{F}_Y)$, such that for all $\omega \in Y$ and $E \in \mathcal{F}_Y$: $[t_i(\omega)] \subseteq E$ implies $t_i(\omega)(E) = 1$, where $[t_i(\omega)] := \{\omega' \in Y : t_i(\omega') = t_i(\omega)\}$,

Proposition 1 *If $\underline{\Omega}$ is a unawareness belief structure, then $F(\underline{\Omega})$ is a standard type-space. Moreover, it has the following property: For every $p > 0$, measet $E \in \mathcal{F}$ and $i \in I$: $\{\omega \in \Omega : t_i(\omega)(E) \geq p\} = \{\omega \in \Omega : t_i^F(\omega)(E) \geq p\}$ (and hence $\{\omega \in \Omega : t_i(\omega)(E) > p\} = \{\omega \in \Omega : t_i^F(\omega)(E) > p\}$.)*

Proof. We only have to show:

1. $t_i^F : \Omega \longrightarrow \Delta(\Omega, \mathcal{F})$ is measurable, where $\Delta(\Omega, \mathcal{F})$ is endowed with the sigma-algebra generated by sets $\{\mu \in \Delta(\Omega, \mathcal{F}) : \mu(E) \geq p\}$ for $p \in [0, 1]$ and $E \in \mathcal{F}$.
2. For all $\omega \in \Omega$, $i \in I$, and $E \in \mathcal{F}$: If $[t_i^F(\omega)] = \{\omega' \in \Omega : t_i^F(\omega') = t_i^F(\omega)\} \subseteq E$, then $t_i^F(\omega)(E) = 1$.

But both properties follow directly from the respective properties in the unawareness belief structure $\underline{\Omega}$. □

Proposition 2 *Extend the definition of the belief operator, Definition 5, in Heifetz, Meier and Schipper (2007), to the collection of all measets $\mathcal{F}$ in $\underline{\Omega}$. Properties (0) to (v) of Proposition 7 in Heifetz, Meier and Schipper (2007) extend to measets. More formally, let E and F be measets, $\{E_l\}_{l=1,2,\ldots}$ be an at most countable collection of measets, and $p, q \in [0, 1]$. The following properties of belief obtain:*

(o) $B_i^p(E) \subseteq B_i^q(E)$, *for* $q \leq p$,

(i) Necessitation: $B_i^1(\Omega) = \Omega$,

(ii) Additivity: $B_i^q(E) \subseteq \Omega \setminus B_i^p(\Omega \setminus E)$, *for* $p + q > 1$,

(iiia) $B_i^p\left(\bigcap_{l=1}^{\infty} E_l\right) \subseteq \bigcap_{l=1}^{\infty} B_i^p(E_l)$,

(iiib) for any decreasing sequence of measets $\{E_l\}_{l=1}^{\infty}$, $B_i^p\left(\bigcap_{l=1}^{\infty} E_l\right) = \bigcap_{l=1}^{\infty} B_i^p(E_l)$,

(iiic) $B_i^1\left(\bigcap_{l=1}^{\infty} E_l\right) = \bigcap_{l=1}^{\infty} B_i^1(E_l)$,

(iv) Monotonicity: $E \subseteq F$ *implies* $B_i^p(E) \subseteq B_i^p(F)$,

(v) Introspection: $B_i^p(E) \subseteq B_i^1 B_i^p(E)$.

Proof. The proof is analogous to the proof of Proposition 7 in Heifetz, Meier and Schipper (2007), making use of Proposition 1 above. $\square$

Extend the definitions of mutual belief and common certainty (Definition 14 in Heifetz, Meier and Schipper, 2007) to the collection of all measets $\mathcal{F}$ in $\underline{\Omega}$. We have the following standard characterization of common certainty:

Definition 6 *An event E is evident if for each $i \in I$, $E \subseteq B_i^1(E)$.*

Proposition 3 *For every measet $F \in \mathcal{F}$,*

(i) $CB^1(F)$ is evident, that is $CB^1(F) \subseteq B_i^1(CB^1(F))$ for all $i \in I$,

(ii) there exists an evident measet E such that $\omega \in E$ and $E \subseteq B_i^1(F)$ for all $i \in I$ if and only if $\omega \in CB^1(F)$.

The proof is analogous to Proposition 3 in Monderer and Samet (1989) for a standard state-space and thus omitted.

There is even a more fundamental difference between an unawareness belief structure and its flattened structure. Namely, there are unawareness belief structures with a non-degenerate common prior on the upmost state-space while the flattened structures do not have a non-degenerate common prior. As an example consider again the unawareness belief structure illustrated in Figure 3 in Heifetz, Meier and Schipper (2007). It has a non-degenerate common prior on the upmost state-space S. Yet, the flattened structure is as follows:

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9	ω_{10}	ω_{11}	ω_{12}	ω_{13}	ω_{14}
Solid	$\frac{1}{6}$	$\frac{3}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\vert 0$	0	$\frac{1}{3}$	$\frac{2}{3}$	0	0	0	$0 \vert$	$\frac{1}{3}$	$\frac{2}{3}$
Dashed	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\vert \frac{1}{3}$	$\frac{2}{3}$	0	0	0	0	0	0

Clearly, there is no non-degenerate common prior in the flattened structure. A non-degenerate common prior in the flattened structure would have to give positive probability to every cell of every player. But then such a prior would have to give a positive probability to state ω_5 because it has a positive posterior for the dashed player, and at the same time probability zero because it has posterior zero for the solid player.

4 Proof of Theorem 1

Let $\underline{\Omega}$ be a topological unawareness belief structure and P a non-degenerate common prior. We have to show that there is no evident measet $E \in \mathcal{F}$ such that $\tilde{\omega} \in E$ and

$$\int_{\Omega} v(\cdot) d(t_1(\omega))(\cdot) \leq x_1 < x_2 \leq \int_{\Omega} v(\cdot) d(t_2(\omega))(\cdot)$$

for all $\omega \in E$.

We require following lemmata:

Lemma 1 *Let $\underline{\Omega}$ be a topological unawareness belief structure, $v : \Omega \longrightarrow \mathbb{R}$ be a continuous random variable, and $x \in \mathbb{R}$. Then $\left\{\omega \in \Omega : \int_{\Omega} v(\cdot)d(t_i(\omega))(\cdot) \geq x\right\}$ and $\left\{\omega \in \Omega : \int_{\Omega} v(\cdot)d(t_i(\omega))(\cdot) \leq x\right\}$ are closed subsets of Ω.*[2]

PROOF OF LEMMA. Since for every $S \in \mathcal{S}$, the topology on $\Delta(S)$ coincides with the weak* topology and since in particular, $v : S \longrightarrow \mathbb{R}$ is continuous, $\left\{\mu \in \Delta(S) : \int_S v(\cdot)d\mu(\cdot) < x\right\}$ is open in $\Delta(S)$. Hence $\left\{\mu \in \bigcup_{S\in\mathcal{S}} \Delta(S) : \int_S v(\cdot)d\nu(\cdot) < x\right\}$ is open in $\bigcup_{S\in\mathcal{S}} \Delta(S)$.

By the continuity of $t_i : \Omega \longrightarrow \bigcup_{S\in\mathcal{S}} \Delta(S)$, it follows that $\left\{\omega \in \Omega : \int_{\Omega} v(\cdot)d(t_i(\omega))(\cdot) < x\right\}$ is open in Ω and hence it's relative complement with respect to Ω, $\left\{\omega \in \Omega : \int_{\Omega} v(\cdot)d(t_i(\omega))(\cdot) \geq x\right\}$ is closed in Ω. $\square$

Lemma 2 *Let $\underline{\Omega}$ be a topological unawareness belief structure. Let E be a closed subset of Ω. Then $CB^1(E)$ is a closed subset of Ω.*

PROOF OF LEMMA. The relative complement of E with respect of Ω, $\Omega \setminus E$, is open, and hence for every $S \in \mathcal{S}$, $(\Omega\setminus E)\cap S = S\setminus(E\cap S)$ is open in S. Therefore $\{\mu \in \Delta(S) : \mu(S\setminus(E\cap S)) > 0\}$ is open. It follows that $\bigcup_{S\in\mathcal{S}} \{\mu \in \Delta(S) : \mu(S \setminus (E \cap S)) > 0\}$ is open. Hence for every $i \in I$, $\left\{\omega \in \Omega : t_i(\omega) \in \bigcup_{S\in\mathcal{S}} \{\mu \in \Delta(S) : \mu(S \setminus (E \cap S)) > 0\}\right\}$ is open. It follows that it's relative complement with respect to Ω, $B_i^1(E) = \left\{\omega \in \Omega : t_i(\omega) \in \bigcup_{S\in\mathcal{S}}\{\mu \in \Delta(S) : \mu(E \cap S) = 1\}\right\}$ is closed. Since an arbitrary intersection of closed sets is closed, the Lemma follows by induction. $\square$

Lemma 3 *Let $\underline{\Omega}$ be a topological unawareness belief structure. Then for every $\omega \in \Omega$, every state-space $S \in \mathcal{S}$ and every player $i \in I$, the set $\{\omega' \in \Omega : t_i(\omega') = t_i(\omega)\} \cap S$ is closed in S.*

PROOF OF LEMMA. Since $\Delta(S_{t_i(\omega)})$ is the set of regular Borel probability measures on $S_{t_i(\omega)}$ endowed with the topology of weak convergence, $\{t_i(\omega)\}$ is closed in $\Delta(S_{t_i(\omega)})$, and hence $\{t_i(\omega)\}$ is closed in $\bigcup_{S\in\mathcal{S}} \Delta(S_{t_i(\omega)})$. Therefore, by continuity of t_i, $t_i^{-1}(\{t_i(\omega)\}) = [t_i(\omega)]$ is closed in Ω. Hence, $[t_i(\omega)] \cap S$ is closed in S. $\square$

Lemma 4 *Let $\underline{\Omega}$ be a topological unawareness belief structure. Let P^S be a non-degenerate (common) prior on the some state-space S, and let $\omega \in S$ such that $t_i(\omega) \in \Delta(S)$. Then, for every $E \in \mathcal{F}_S$, we do have $t_i(\omega)(E) = t_i(\omega)(E \cap [t_i(\omega)]) = \frac{P^S(E\cap[t_i(\omega)])}{P^S(S\cap[t_i(\omega)])}$.*

PROOF. We have $t_i(\omega)(S \cap [t_i(\omega)]) = 1$ and hence $t_i(\omega)(E) = t_i(\omega)(E \cap S \cap [t_i(\omega)]) = t_i(\omega)(S \cap [t_i(\omega)])$. Since P^S is non-degenerate, we do have $P^S(S \cap [t_i(\omega)]) > 0$.

Since $S((E \cap [t_i(\omega)])^{\uparrow}) = S$ and since $\omega' \in [t_i(\omega)]$ implies $t_i(\omega') \in \Delta(S)$, we do have $(E \cap [t_i(\omega)])^{\uparrow} \cap A_i((E \cap [t_i(\omega)])^{\uparrow}) = (E \cap [t_i(\omega)])^{\uparrow}$. We also have $(S \cap [t_i(\omega)])^{\uparrow} \subseteq A_i(S^{\uparrow}) =$

[2]Note that we abuse notation and write $\int_{\Omega} v(\cdot)d(t_i(\omega))(\cdot)$ instead of $\int_{S_{t_i(\omega)}} v(\cdot)d(t_i(\omega))(\cdot)$.

$A_i((E \cap [t_i(\omega)])^{\uparrow})$. Last equality follows from weak necessitation. We have - by the definition of a common prior - the following (with our abuse of notation):

$$\begin{aligned} P^S(E \cap [t_i(\omega)]) &= \int_{S \cap A_i((E \cap [t_i(\omega)])^{\uparrow})} t_i(\cdot)(E \cap [t_i(\omega)]) dP^S(\cdot) \\ &= \int_{S \cap [t_i(\omega)]} t_i(\cdot)(E \cap [t_i(\omega)]) dP^S(\cdot) \\ &\quad + \int_{(S \cap A_i(S^{\uparrow})) \setminus (S \cap [t_i(\omega)])} t_i(\cdot)(E \cap [t_i(\omega)]) dP^S(\cdot) \end{aligned}$$

But if $\omega' \in (S \cap A_i((E \cap [t_i(\omega)])^{\uparrow})) \setminus (S \cap [t_i(\omega)])$, then $t_i(\omega')(E \cap [t_i(\omega)]) = 0$, and hence, we have

$$\begin{aligned} P^S(E \cap [t_i(\omega)]) &= \int_{S \cap [t_i(\omega)]} t_i(\cdot)(E \cap [t_i(\omega)]) dP^S(\cdot) \\ &= t_i(\omega)(E \cap [t_i(\omega)]) \int_{S \cap [t_i(\omega)]} 1 dP^S(\cdot) \\ &= t_i(\omega)(E \cap [t_i(\omega)]) P^S(S \cap [t_i(\omega)]). \end{aligned}$$

Since $P^S(S \cap [t_i(\omega)]) > 0$, it follows that $t_i(\omega)(E \cap [t_i(\omega)]) = \frac{P^S(E \cap [t_i(\omega)])}{P^S(S \cap [t_i(\omega)])}$. □

Proof of the Theorem: Suppose by contradiction, that there are $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and a continuous random variable $v : \Omega \longrightarrow \mathbb{R}$ such that $CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \neq \emptyset$, where

$$\begin{aligned} E_1^{\leq x_1} &:= \left\{ \omega \in \Omega : \int_{S_{t_1(\omega)}} v(\cdot) d(t_1(\omega))(\cdot) \leq x_1 \right\}, \text{ and} \\ E_2^{\geq x_2} &:= \left\{ \omega \in \Omega : \int_{S_{t_2(\omega)}} v(\cdot) d(t_2(\omega))(\cdot) \geq x_2 \right\}. \end{aligned}$$

Let S be a $\preceq$-minimal state-space with the property that $S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \neq \emptyset$.

By Proposition 2 we have $CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \subseteq B_i^1(CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))$ for $i = 1, 2$. This implies that for each $\omega \in S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ and $i = 1, 2$, we have $t_i(\omega)(CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})) = 1$, which by the minimality of S implies that $t_i(\omega) \in \Delta(S)$ and $t_i(\omega)(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})) = 1$.

By Lemma 2, $S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))$ is closed in S. Therefore it is easy to verify that if flattened, $F(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))$, that is $S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ with the induced structure, is a standard topological type-space (as in Heifetz, 2006). Since each $\omega \in S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$, we have $t_i(\omega)(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})) = 1$ for $i = 1, 2$.

Since P^S is a non-degenerate prior on S, we have that $P^S(S \cap [t_i(\omega)]) > 0$, for each $\omega \in S$.

For $\omega \in S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ we also have $t_i(\omega)(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \cap [t_i(\omega)]) = 1$, and by Lemma 4, we have $t_i(\omega)(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \cap [t_i(\omega)]) = \frac{P^S(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \cap [t_i(\omega)])}{P^S(S \cap [t_i(\omega)])}$.

Hence, since $P^S(S \cap [t_i(\omega)]) > 0$, it follows that $P^S(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \cap [t_i(\omega)]) = P^S(S \cap [t_i(\omega)]) > 0$. It follows that $P^S(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})) > 0$. Therefore it is easy to check that $\frac{P^S(\cdot)}{P^S(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))}$ is a common prior on $F(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))$.

Claim: Let $\omega \in CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \cap S$. Then $\int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) d(t_1(\omega))(\cdot) \leq x_1$ and $\int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) d(t_2(\omega))(\cdot) \geq x_2$.

We prove the second inequality, the first is analogous to the second one. We know already that $t_2(\omega) \in \Delta(S)$. By the definitions $\omega \in S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ implies $\omega \in S \cap B_2^1(E_2^{\geq x_2})$, and therefore $t_2(\omega)([t_2(\omega)] \cap E_2^{\geq x_2} \cap S) = 1$. It follows that $[t_2(\omega)] \cap E_2^{\geq x_2} \cap S$ is non-empty. Let $\omega' \in [t_2(\omega)] \cap E_2^{\geq x_2} \cap S$. Then we have $\int_S v(\cdot) d(t_2(\omega'))(\cdot) \geq x_2$. But we have $t_2(\omega) = t_2(\omega')$ and therefore $\int_S v(\cdot) d(t_2(\omega))(\cdot) \geq x_2$.

Since S is compact and $v : S \longrightarrow \mathbb{R}$ is continuous, there is a $\bar{v} \in \mathbb{R}$ such that $|v(\tilde{\omega})| \leq \bar{v}$ for all $\tilde{\omega} \in S$.

Since $t_2(\omega)(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})) = 1$, we have

$$\begin{aligned} \left| \int_{S \setminus (S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))} v(\cdot) d(t_2(\omega))(\cdot) \right| &\leq \bar{v} \int_{S \setminus (S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))} 1 d(t_2(\omega))(\cdot) \\ &= \bar{v} \quad t_2(\omega)(S \setminus (S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))) \\ &= 0. \end{aligned}$$

Hence, we have

$$\int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) d(t_2(\omega))(\cdot) = \int_S v(\cdot) d(t_2(\omega))(\cdot) \geq x_2$$

and this finishes the proof of the claim.

It follows that we have found a standard topological type-space $S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ in the sense of Heifetz (2006) with a common prior and a continuous random variable $v : S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \longrightarrow \mathbb{R}$ such that

$$\int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) d(t_1(\omega))(\cdot) \leq x_1 < x_2 \leq \int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) d(t_2(\omega))(\cdot).$$

Note that if we replace $v(\cdot)$ by $v(\cdot) - \frac{x_1 + x_2}{2}$, we get

$$\int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) - \frac{x_1 + x_2}{2} d(t_1(\omega))(\cdot) < 0 < \int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) - \frac{x_1 + x_2}{2} d(t_2(\omega))(\cdot).$$

But this is a contradiction to Feinberg's (2000) Theorem (Proposition 1 in Heifetz, 2006). Hence this completes the proof of the theorem. □

5 Equilibria in the Flattened Structure

The Flattened Game: Given a Bayesian game with unawareness of events and (possibly) action $\Gamma(\underline{\Omega})$, we can associate a standard Bayesian game $F(\Gamma(\underline{\Omega}))$ played on a standard type-space (with possibly allowing for varying action sets of the players across different types) in the following manner:

If $\Gamma(\underline{\Omega}) = \langle \underline{\Omega}, (M_i)_{i \in I}, (\mathcal{M}_i)_{i \in I}, (u_i)_{i \in I} \rangle$, where $\underline{\Omega} = \langle \mathcal{S}, (r_{S_\beta}^{S_\alpha})_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I} \rangle$ is a unawareness belief structure, then set $F(\Gamma(\underline{\Omega})) := \langle F(\underline{\Omega}), (M_i)_{i \in I}, (\mathcal{M}_i)_{i \in I}, (u_i)_{i \in I} \rangle$, where $F(\underline{\Omega})$ is the flattened structure associated with $\underline{\Omega}$, and $(M_i)_{i \in I}$, $(\mathcal{M}_i)_{i \in I}$, and $(u_i)_{i \in I}$ remain unchanged.

Remark 3 *Since the strategy sets and the utility functions remain unchanged, we have that any strategy profile is a Bayesian unawareness equilibrium in* $\Gamma(\underline{\Omega})$ *if and only if it is a Bayesian equilibrium in* $F(\Gamma(\underline{\Omega}))$.

We view flattening the game as a purely technical procedure since there is no natural interpretation of a flattened game. For instance, in the flattened game we can have types of players who are certain of their set of actions but consider it possible that they have a larger set of actions even though they don't have a larger set of actions. This leads to serious conceptional problems if a player would choose such an action.[3] Note also that equilibria of flattened games can not be interpreted as equilibria under unawareness. Since the flattened structure is a standard type-space, the Dekel-Lipman-Modica-Rustichini (1998) critique applies. Hence unawareness is trivial in the flattened structure.

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